JRF (Quality, Reliability & Operations Research): 2015
INDIAN STATISTICAL INSTITUTE

INSTRUCTIONS

The test is divided into two sessions (i) Forenoon session and (ii) Afternoon session. Each session is of two hours duration. For the forenoon session question paper, the test code is **MMA** and for the afternoon session question paper, the test code is **QRB**. Candidates appearing for JRF (QROR) should verify and ensure that they are answering the right question paper.

For test **MMA**, see a different Booklet. For test **QRB**, refer to this Booklet only.

The test **QRB** is of short answer type. It has two groups. A candidate has to answer **six questions** choosing **at least two from each group**.

OUTLINE OF THE SYLLABUS FOR QRB

The syllabus for JRF (QROR) will include the following subject-groups:

1 **Group A**

- i) Mathematics
- ii) Probability and Statistics

2 **Group B**

- i) Operations Research
- ii) Reliability
- iii) Statistical Quality Control and Quality Management

A broad coverage for each of the above subjects is given below.

i) **Mathematics**: Set theory; Algebra; Real analysis; Calculus; Vectors and matrices.

ii) **Probability and Statistics**: Elementary probability and distribution theory; Bivariate distributions; Multivariate normal distribution; Linear models and regression; Estimation; Test of hypothesis; Design of experiments (block design, full and fractional factorial designs); Markov chain.

iii) **Operations Research**: Linear programming (basic theory, simplex algorithm and its variants, duality theory, transportation and assignment problem); Non-linear programming-basic theory; Generalized convexity; Game theory (bi-matrix game including zero-sum game).

iv) **Reliability**: Coherent systems and system reliability; Hazard function; Failure time distribution; Censoring schemes; Estimation and testing in reliability; Replacement models; Repairable system.

v) **Statistical Quality Control and Quality Management**:

Statistical Quality Control: Statistical process control - attribute and variable control charts; Multivariate control chart; Control chart with memory (CUSUM, EWMA etc.) (univariate only); Process capability analysis; Acceptance sampling; Engineering Process Control.

Quality Management: Quality concepts - different views, dimensions and theories of quality management, concepts of service quality and different models like SERVQUAL and SERVPREF; distinction between product and service quality; Organizational design and its impact on quality; Concepts of customers - internal and external, individual and institutional; definition and measurement of customer satisfaction for individual as well as industrial customers; Quality Management models - MBNQA and EFQM; Costs of quality; six sigma and its variants

Sample Questions for Mathematics (Group A)

1. Let $f(x) = 3x^4 + 4x^3 - 2x^2 - 4x + 5$ for $x \in [-1, 1]$. Find the maximum value of f in the interval [-1, 1]. Is the function f convex on the interval [-1, 1]?

2. Let $a_n > 0$ for $n \geq 1$. Suppose $\sum_{n=1}^{\infty} a_n$ is divergent. Show that $\sum_{n=1}^{\infty} \frac{a_n}{1+a_n}$ is divergent.

3. Let p be a prime number greater than 3. Show that $p^2 - 1$ is divisible by 24.

4. Let f be a real valued thrice differentiable function defined by $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$, where x is real. Find the value of $f(2)$.

5. Consider functions from $A = \{0, 1, 2\}$ to $B = \{0, 1, 2, \ldots, 7\}$. Find the number of functions f satisfying $f(p) < f(q)$ for $p < q$, and hence the number of functions satisfying $f(p) \leq f(q)$ for $p < q$.

6. Find the total number of functions $f : \{1, 2, \ldots, 7\} \rightarrow \{-1, 1\}$ satisfying
$$\sum_{i=1}^{7} f(i) > 1.$$

7. Let B be a nonsingular matrix of order n and $D = B'B$. Show that if $x'Dx = 0$, then $x = 0$.

8. Let $f : Z \rightarrow R$ where Z is the set of all integers and R is the set of all real numbers such that $f(n) + f(n-1) + f(n-2) = 0$ for all $n \in Z$ and $f(0) = 1$. Find the value of $\sum_{k=1}^{101} f(k)$.

9. Consider the set
$A = \{u = 15x + 21y : x \text{ and } y \text{ are integers}, 0 \leq u \leq 4000\}$.
Find the number of elements in A.

10. $\{x_n\}$ and $\{y_n\}$ are two sequences of real numbers such that $2x_n^2 + x_n y_n + 2y_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Show that $x_n \rightarrow 0$ and $y_n \rightarrow 0$.

11. Let A be a 3×3 real matrix such that $A^2 = 0$. What is the maximum possible rank of A?

12. Prove that if $a_1, a_2, ..., a_n$ are positive numbers, then

$$(a_1 + a_2 + \cdots + a_n)\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) \geq n^2.$$

13. A point is randomly chosen on each side of a unit square. Let a, b, c and d be the sides (lengths) of the quadrilateral formed by these four points. Show that

    (i) $2 \leq a^2 + b^2 + c^2 + d^2 \leq 4$

    (ii) $2\sqrt{2} \leq a + b + c + d \leq 4$.

14. Consider the sequence $\{(5^n + 4^n)^{\frac{1}{n}},\ n = 1, 2, \ldots\}$. Show that the sequence is bounded and decreasing.

15. Let $A \in R^{m \times n}$ and $D = AA^t$. Define

$$\tilde{A} = \begin{bmatrix} D & -A \\ A^t & 0 \end{bmatrix}.$$

Show that $x^t \tilde{A} x \geq 0\ \forall x \in R^{m+n}$.

16. Let Q denote the set of rational numbers. For given constants a, b, c, define the function f on real numbers:

$$f(x) = \begin{cases} ax & \text{if } x \in Q \\ c - bx & \text{if } x \notin Q. \end{cases}$$

Derive necessary and sufficient conditions on the constants a, b, c for each of the following cases:

   i) f is a continuous function,

   ii) f is continuous exactly at one point and

   iii) f is nowhere continuous.

17. Let

$$g(t) = \int_0^t \frac{1}{x} \sin\left(\frac{tx}{\pi}\right) dx, \quad \text{for } t > 0.$$

Find $g'(\pi)$.

Sample Questions for Probability and Statistics(Group A)

1. Suppose that two positive numbers are chosen randomly from 1 to 50. What is the probability that their difference is divisible by 3?

2. A secretary goes to work following one of the three routes A, B and C. Her choice of route is independent of weather. If it rains, the probability of arriving late following A, B and C are 0.06, 0.15 and 0.12, respectively. The corresponding probabilities if it does not rain are 0.05, 0.10 and 0.15. Assume that, on an average, one in every four days is rainy.

   i) Given that she arrives late on a day without rain, what is the probability that she took route C?

   ii) Given that she arrives late on a day, what is the probability that it was a rainy day?

3. The probability p_k that a family has k children is given by

$$p_k = \begin{cases} a & \text{for } k = 0, 1; \\ (1-2a)^{-(k-1)} & \text{for } k \geq 2, \end{cases}$$

   where $a < 1/2$. Assuming that the probability of a boy child is the same as that of a girl child in a family, compute the conditional probability that the family has only two children given that the family has two boys.

4. Suppose die A has 4 red faces and 2 green faces while die B has 2 red faces and 4 green faces. Assume that both the dice are unbiased. An experiment is started with the toss of an unbiased coin. If the toss results in a Head, then die A is rolled repeatedly while if the toss of the coin results in a Tail, then die B is rolled repeatedly. For $k = 1, 2, 3, \ldots$, define X_k to be the indicator random variable such that X_k takes the value 1 if the k-th roll of the die results in a red face, and takes the value 0 otherwise.

   i) Find the probability mass function of X_k.

ii) Calculate the correlation between X_1 and X_7.

5. Consider a Markov Chain $\{X_n, n = 0, 1, 2, \ldots\}$ with state space $S = \{0, 1, 2, \ldots\}$ such that

$$P\left[X_n = j | X_{n-1} = i\right] = \begin{cases} p & \text{if } j = i + 1; \\ 1 - p & \text{if } j = i. \end{cases}$$

   i) Derive $P\left[X_n = 3 | X_0 = 0\right]$, for $n = 0, 1, 2, \ldots$.
   ii) Comment on the irreducibility of the Markov chain with proper justification.
   iii) Comment on the stationary distribution of the Markov Chain with proper justification.

6. Let $\{X_n, n = 0, 1, 2, \ldots\}$ be a Markov chain with state space $S = \{0, 1, 2, 3\}$ and transition matrix

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{2}{3} \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   i) Calculate $P[X_3 = 1 | X_0 = 0]$.
   ii) Check whether the chain is irreducible.

7. Let (X, Y) have the joint probability density function given by

$$f(x, y) = \begin{cases} 24xy & \text{for } 0 < x, y, x + y < 1 \\ 0 & \text{otherwise.} \end{cases}$$

   Find the correlation coefficient between X and Y.

8. Suppose $(Y, Z)' \sim N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and the conditional distribution of Ygiven $Z = z$ is $N(-z, 1)$. Also it is given that $E(Z | Y = y) = -\frac{1}{3} - \frac{1}{3}y$. Then determine $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$.

9. Let $X_{(1)}$, $X_{(2)}$ and $X_{(3)}$ be the order statistics of a random sample of size 3 from

$$f_\theta(x) = \begin{cases} \dfrac{1}{\theta}, & \text{if } 0 \leq x \leq \theta \\ 0, & \text{otherwise,} \end{cases}$$

for $0 < \theta < \infty$. Let $T_1 = \alpha_1 X_{(1)}$, $T_2 = \alpha_2 X_{(2)}$ and $T_3 = \alpha_3 X_{(3)}$ be unbiased estimators of θ where α_1, α_2 and α_3 are suitable positive numbers. Find the values of α_1, α_2 and α_3. Compare the efficiencies of T_1, T_2 and T_3.

10. Consider independent random variables $U_1 \sim N(\theta, 1)$, $U_2 \sim N(\theta, 1)$ and $U_3 \sim N(0, 1)$. Suppose these variables are unobservable, but $X = U_1 + U_2$ and $Y = U_2 + U_3$ are observed.

    (a) Find the best linear unbiased estimator (BLUE) of θ based on X and Y.

    (b) Find the variance of the BLUE obtained in part (a).

    (c) If independent paired samples (X_i, Y_i), $i = 1, \ldots, n$ are generated as above, find the BLUE of θ based on the $2n$ observations.

    (d) Provide an exact confidence interval for θ with coverage probability 0.95, based on the observations mentioned in part (c).

11. Consider a random sample of size 1 from the population with density

$$f(x; \theta) = 2(\theta - x)/\theta^2,\ 0 \leq x < \theta,\ \theta > 0.$$

    i) Obtain the maximum likelihood estimate of θ (by checking if it maximizes the likelihood).

    ii) Find an unbiased estimator of θ.

12. Consider the linear model

$$Y_i = \beta \frac{i}{n} + \epsilon_i,\ i = 1, \ldots, n,$$

where β is the unknown regression parameter of interest and ϵ_i's are independent error variables satisfying $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2$ (unknown). Find Best Linear Unbiased Estimator of β.

13. Consider the following fixed effect linear model:

$$\begin{aligned} y_1 &= \alpha + \epsilon_1 \\ y_2 &= y_1 + \alpha - \beta + \epsilon_2 \\ y_3 &= y_2 + \alpha + \epsilon_3 \\ y_4 &= y_3 + \alpha + \beta + \epsilon_4 \\ y_5 &= y_4 + \alpha + 2\beta + \epsilon_5, \end{aligned}$$

where $y_1, y_2, \ldots, y_5$ are the observations; α, β are the parameters and ϵ_i's are uncorrelated random variables with $E(\epsilon_i) = 0$ and $Var(\epsilon_i) = \sigma^2, i = 1\,(1)\,5$. Find the least square estimates of α and β based on the observations $y_1 = 1, y_2 = 2, y_3 = 3, y_4 = 4, y_5 = 6$.

14. The average yield of grain in an experiment with treatment A is claimed to be greater than that with treatment B. Describe a statistical test for this claim, based on yield data from n_A and n_B experiments with treatment A and B, respectively. Clearly specify the null and alternative hypotheses and the underlying assumptions needed.

15. Let $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_5$ be a random sample from $N_2(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, where

$$\boldsymbol{\Sigma} = \begin{pmatrix} 3 & 0.1 \\ 0.1 & 1 \end{pmatrix}.$$

Suppose that the sample mean is $\bar{X} = (1, 0)'$. Test the hypothesis

$$H_0 : \boldsymbol{\mu} = (0, 0)' \quad \text{Vs} \quad H_1 : \boldsymbol{\mu} \neq (0, 0)'$$

at 5% level of significance. [Given that, $\chi^2_{0.05,2} = 5.99$].

16. Fold over a 2^{5-2} design to construct a 2^{6-2} design. Write the complete defining relation of the resulting design. What is its resolution? Is the resulting design a minimal design?

Sample Questions for Operations Research (Group B)

1. Consider the following problem:

$$\text{minimize } c'x, \text{ subject to } Ax \geq b, x \geq 0,$$

where $A \in R^{m\times n}, c \in R^n$ and $b \in R^m$.

   i) Show that the optimal solution to this linear programming problem is equivalent to solving a system of equations in non-negative variables.

   ii) If the optimal solution is not unique then show that there cannot be finitely many optimal solutions to the linear programming problem.

2. Show that the set of all feasible solutions to a linear programming problem is a convex set.

3. Consider the problem:

$$\text{minimize } \frac{p^t x + \alpha}{q^t x + \beta}, \; q^t x + \beta \neq 0$$

$$\begin{aligned} \text{subject to } Ax &= b \\ x &\geq 0, \end{aligned}$$

where $p, q \in R^n$, $b \in R^m$, $A \in R^{m\times n}$ and $\alpha, \beta \in R$.
Formulate the above problem as linear programming problem.

4. (a) State the fundamental theorem of duality.

   (b) Using the above theorem, show, without actually solving it, that the following problem has an optimal solution:

$$\begin{aligned} &\text{Maximize } 5x_1 - 3x_2 + 9x_3 \\ &\text{subject to} \\ &\quad 2x_1 - 3x_2 + 5x_3 \leq 10 \\ &\quad -x_1 + 2x_2 + x_3 \leq 11 \\ &\quad 4x_1 - 2x_2 + 3x_3 \leq 12 \\ &\quad x_1, x_2, x_3 \geq 0. \end{aligned}$$

5. Consider the optimization problem:

$$\text{Minimize } c^t x \text{ subject to } e^t x = 0,$$

where $c,\ x,\ e$ are column vectors in $\mathbf{R}^n$, e is the vector of 1s and c is a fixed vector.

   (a) Formulate the problem as a linear programming problem with non-negative variables.

   (b) State the condition under which the problem has an optimal solution.

6. If the i^{th} row of the payoff matrix of an $m \times n$ rectangular game be strictly dominated by a convex combination of the other rows of the matrix, then show that the deletion of the i^{th} row from the matrix does not affect the set of optimal strategies for the row player.

7. Let $f : R^n \to R$ be pseudoconvex at $\bar{\mathbf{x}}$. Then show that $\bar{\mathbf{x}}$ is a global minimum if and only if $\nabla f(\bar{\mathbf{x}}) = 0$.

8. Show that a balanced transportation problem always has a feasible solution.

9. Let $A \in R^{m \times n}$. Show that exactly one of the following two systems has a solution:

$$\begin{array}{ll} \text{System 1 :} & Ax < 0 \text{ for some } x \in R^n \\ \text{System 2 :} & A^t y = 0,\ y \geq 0 \text{ for some } y \in R^m. \end{array}$$

10. Solve the following problem:

$$\begin{aligned} \text{maximize } f(x_1, x_2, x_3) &= x_1 x_2^2 x_3 \\ \text{subject to } x_1 + x_2 + x_3^2 &= k, \end{aligned}$$

where $k > 0$ is fixed and x_1, x_2, x_3 are positive real numbers.

11. Find the global minimizers (if any) for the following functions:

$$\begin{array}{l} i)\ f(x,y) = e^{x-y} + e^{y-x}, \text{ and} \\ ii)\ f(x,y) = e^{x-y} + e^{x+y}. \end{array}$$

Sample Questions for Reliability (Group B)

1. The UN security council consists of five permanent members having 7 points each and ten temporary members having 1 point each. In order to get a resolution passed, at least 39 points are needed in favour. Present this problem (of getting a resolution passed) as a coherent system with 15 components and obtain the structural importance of the components. Find a non-trivial modular decomposition of the system.

2. Prove that the min path sets of a coherent system are the min cut sets of its dual and vice versa.

3. Consider a coherent system with four independent components 1, 2, 3 and 4, in which the min path sets are $\{1, 3\}$, $\{2, 3\}$ and $\{4\}$.

   (a) Write down the structure function of the system.

   (b) Write down the min cut sets of the system.

   (c) If the lifetime of the ith component follows the uniform distribution over $[0, i]$ for $i=$ 1, 2, 3, 4, find the reliability of the system at time $t > 0$.

   (d) Prove that the reliability of the above system at time t is less than or equal to

   $$1 - \prod_{i=1}^{4} \left[\frac{\min\{t, i\}}{i} I(t > 0) \right],$$

   where $I(\cdot)$ denotes the indicator function.

4. Suppose the n components of a parallel system have independent exponential lifetimes with failure rates as $\lambda_1, \cdots, \lambda_n$. Compute the system reliability function and the mean system lifetime.

5. Suppose that two independent systems, with identical and exponentially distributed lifetime with failure rate 3×10^{-7} per hour, are either to be placed in active parallel or a stand-by configuration. For $t = 7$ years, derive the reliability gain of one over the other configuration.

6. Consider the Weibull distribution with reliability function

$$R(t) = e^{-\alpha t^{\beta}}, t > 0, \alpha > 0, \beta > 0.$$

   i) When does the distribution have decreasing failure rate?

   ii) Obtain an expression, possibly in the form of an integral, for the mean remaining life function for this distribution.

7. Consider a series system with two independent components each with strength having exponential distribution with parameter α. The system is subjected to a random stress following exponential distribution with parameter β. A component fails when the stress exceeds its strength. Find the reliability of the system.

8. Suppose n independent units each having *exponential*(λ) life distribution are put on a life test for a pre-fixed time T_0. Find the distribution of the number of failures during the test. Derive the maximum likelihood estimate of expected number of such failures.

9. Suppose a device has lifetime distribution T with the p.d.f.

$$f(t) = 2\lambda t e^{-\lambda t^2}, \quad t > 0, \lambda > 0.$$

   (i) Find the distribution of T^2.

   (ii) Suppose n identical devices are put on a life test at time $t = 0$. The life test is continued until a pre-fixed number r ($\leq n$) of the devices have failed (the remaining are censored at the r-th failure time). Derive the maximum likelihood estimator (MLE) of λ. Check whether the MLE of λ is unbiased.

10. The hazard rate for an item is given by

$$h(x) = \begin{cases} a, & \text{if } 0 < x \leq x_0 \\ a + b(x - x_0), & \text{if } x > x_0, \end{cases}$$

where a, b, x_0 are positive constants. Derive the reliability function. For known x_0 and based on random right censored lifetime data, write down the likelihood function for estimating a and b.

11. An equipment is replaced by a spare in the event of its failure or having reached the age of T_0, whichever is earlier. Let F be the lifetime distribution of the equipment. Derive the expectation of this replacement time.

12. Consider a one-unit device and suppose that there are n spares stocked at time 0. When the operating unit fails, it is replaced by a spare having independent and identical life distribution given by the density $f(x) = \lambda e^{-\lambda x},\ \lambda > 0, x > 0$. Assume that all the replacements are instantaneous. Derive an expression to determine the minimum value of n such that the probability of uninterrupted service over time interval $(0, t]$ is more than 0.95.

13. Consider a single unit system maintained with independent components having $\exp(\lambda)$ life distribution.

    (a) Prove that the probability of at least n replacements by the first 500 hours of operation is

    $$\int_0^{500} \frac{\lambda^n}{(n-1)!} t^{n-1} e^{-\lambda t} dt.$$

    (b) Find the probability that there are $d_1 (< d)$ replacements in the first 200 hours given that d replacements are observed by the first 500 hours.

    (c) Find the probability that there will be d_2 replacements in the next 200 hours (after observing for 500 hours) given that there are d replacements in the first 500 hours.

    (d) Find the MLE of λ based on the information that there are d replacements in 500 hours and the distribution of this MLE.

Sample Questions for Statistical Quality Control and Quality Management (Group B)

1. For a single sampling plan with curtailed inspection associated with rejection, derive the expression for ASN as a function of product quality.

2. Derive the following properties of the binomial OC function of a single sampling plan with sample size n and acceptance number c at a process average p:

$$B(c+1, n+1, p) - B(c, n, p) = qb(c+1, n, p)$$

where

$$B(c, n, p) = \sum_{x=0}^{c} b(x, n, p) = \sum_{x=0}^{c} \binom{n}{x} p^x (1-p)^{n-x}.$$

3. The thickness (X) of a printed circuit board is an important quality characteristic and $\bar{X}$ chart is being used for monitoring of its mean. A newly recruited quality manager argued that the currently used parameters of the $\bar{X}$ chart, i.e., sample size, sampling interval and control limits may not be appropriate. These should be selected in such a way that the expected net income per production cycle is maximized.
It is observed that the process standard deviation σ remains unchanged. The time required for collection of samples and interpretation of results is proportional to the sample size. Only a single assignable cause of magnitude δ can occur at random according to a Poisson process with an intensity of λ occurrences per hour. Since only a single assignable cause occurs, a fixed time is required for searching and its elimination. Derive the length of a production cycle under the assumption that the process operation is stopped during the search for the assignable cause.

4. For a control chart of a normally distributed quality characteristic if all of the next seven points fall on the same side of the center line, we conclude that the process is out of control. (i)What is

the α risk? (ii)If the mean shifts by one standard deviation and remains there during collection of the next seven samples then find what is the β risk associated with this decision rule.

5. Show that if $\lambda = 2(w+1)$ for the EWMA control chart, the chart is equivalent to a w period moving average control chart in the sense that the control limits are identical for large t.

6. Describe any one method for estimating the process capability indices C_p and C_{pk} when the quality characteristic is a non-normal variable.

7. A process operates at the mid-specification for a variable quality characteristic. Show that the extent of off-specification and the associated loss will be minimum assuming that the natural variation of the process is more than the desired engineering tolerance and the losses incurred per unit on both sides of the specification limits are identical.

8. In designing a fraction nonconforming chart with center line at $\bar{p} = 0.20$ and 3-sigma limits, (i) what sample size (n) is required to yield a positive lower control limit and (ii) what value of n is necessary to obtain a probability of 0.50 for detecting a shift in the process $\bar{p}$ to 0.26?

9. Two different companies - say A and B, engaged in selling biscuits in broadly the same price range have different levels of customer complaints. The number of complaints for A is about 5 per 10000 packs sold whereas the same is about 6 for company B. It is known that the pack sizes and the process of recording complaints are about the same for the two companies. In spite of that a quality management professional claims that the level of satisfaction of the customers for these two companies may not be different. Do you agree? Give one reason for supporting or refuting the proposition using the theory of quality management.

10. A large division of a major consumer goods company is reviewing its quality management practices. The senior management

of the firm is particularly interested in assessing its new product introduction process as this was considered to be the key for competitive success. Two divergent views emerged regarding the quality of this process. One group felt that the process has been quite successful - new products appeared regularly, customer complaints were few, and defective items were not supplied to the trade in numbers that may impact customer confidence. However, the other group felt that the quality of the process is poor as product releases are often delayed and designs often needed change as adhering to manufacturing specifications frequently turn out to be difficult. Explain the divergence of views considering the different views of quality that are likely to have been adopted by the two groups.

11. Consider a hospital that has excellent stuff and equipment but has a rather shabby appearance due to lack of general maintenance. Although the past records show that the rate of cure of patients of the hospital is comparable to the other hospitals in the locality, the trust of general public on the quality of treatment is low. Explain this phenomenon considering that people tend to assess service quality by comparing expectation with performance with respect to various dimensions of service quality.