Mathematics

1. a) What is the maximum value of $x_1+x_2+x_3+x_4$ satisfying the equation:

$$18x_1^2 + 18x_1x_3 + 4x_1x_4 + 10x_2^2 - 6x_2x_3 + 14x_3^2 + 18x_3x_4 + 12x_4^2 = 0.$$

(8)

b) Assume that $f : R \to R$ satisfies: $f(\lambda x+(1-\lambda)y) \leq \lambda f(x)+(1-\lambda)f(y)$ for all $x,\ y$ and for all $\lambda \in [0\ 1]$ interval. Show that for any real r, the set $S = \{x : f(x) \leq r\}$ is a convex set, that is, $\lambda x + (1-\lambda)y \in S$ for all $x,\ y \in S$ and for all $\lambda \in [0\ 1]$ interval. State the converse and prove that it it is not true. (2+1+5=8)

c) Consider the random matrix

$$A = \begin{bmatrix} x_1 & x_4 & 1 \\ x_2 & x_5 & 1 \\ x_3 & x_6 & 1 \end{bmatrix}$$

where each of x_1 to x_6 is from $\{0,\ 1\}$. How many matrices are there of this form? How many of them are nonsingular? (8)

Probability

2. Consider the game of *Tossing Run Race*: Player A has a coin whose probability of Heads is p and Player B has a coin whose probability of Heads is q, where $q < p$. Both simultaneously toss their coins, and the outcome is recorded as $\omega_1\omega_2$, where ω_1 is the result of A's coin and ω_2 is the result of B's coin. If the outcome is HH, A gets one point; if the outcome is TT, then B gets one point, other wise each player gets zero points. After the first toss, the two players toss their coins for a second time independently and the scores are given in the same fashion. At any time during the *race*, if a player's score is 1 point more than the other's, then that player is declared as the winner, and the race stops.

a) Define suitable random variable(s) to model the problem. (8)

b) What is the probability that the at end of 20 tosses, the two scores are same? (8)

c) What is the probability that at the end of 100^{th} toss, Player A wins? (8)

3. a) One hundred people line up to board an airplane. Each has a boarding pass with assigned seat. However, the first person to board has lost his boarding pass and takes a random seat. After that, each person takes her/his assigned seat if it is unoccupied, else one of unoccupied seats at random. What is the probability that the third person in the line gets to sit in her/his assigned seat? (6)

b) A company assesses the status of each of its employees every year and puts the employee in one of the four colour categories - Green, Yellow, Amber and Red. Green means that the employee is in low/low category, that is, the chance of employee leaving the company is low and the impact to the company is also low even if the employee leaves the company. Yellow means that the chance of leaving the company is low but the impact to the company is high if the employee leaves. This is low/high category. Amber stands for high/low, that is, chance of leaving is high but the impact to the company is low. Finally, red stands for high/high. Note that red-coloured employees are critical. Table below presents data on the number of employees that go from one colour to another from previous year to the current year. Make a simplifying assumption that no employee leaves the company and no new employee joins the company. If the same trend continues, what will be the distribution of employees' status four years hence? Suffices to derive the expressions. (12)

Color	Green	Yellow	Amber	Red	Previous year
Green	1026	102	73	14	1215
Yellow	70	290	15	15	390
Amber	27	8	60	2	97
Red	10	7	5	12	34
Current year	1133	407	153	43	1736

Statistics

4. (a) Consider a markov chain with 1000 states with the symmetric one-step transition matrix $P = (p_{ij})$, where p_{ij} is the $(i, j)^{th}$ element of P. Assume that minimum of p_{ij}s is $\frac{1}{1000}$. Find stationary distribution of the markov chain. (12)

   (b) Choose a 2^{7-2} fractional factorial design defined by two four-factor independent interactions. Describe the confounding effects and give the complete alias structure. (12)

Operations Research

5. a) Solve the following problem using graphical method. (8)

$$\texttt{Minimize } 2x - 3y$$

$$\texttt{subject to}$$

$$\begin{aligned} -3x + 4y &\leq 4, \\ x + \quad y &\leq 6, \\ 2x - \quad y &\leq 6, \\ x, y &\geq 0. \end{aligned}$$

   b) Show that the optimization problem below has an optimum solution. (8)

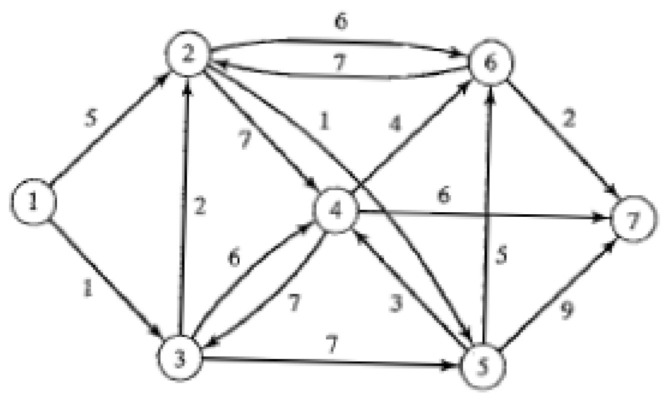
$$\texttt{Minimize } 3x + 4y$$

$$\texttt{subject to}$$

$$x^2y^3 = 600,$$

$$x, y \geq 0.$$

c) Consider the network diagram below. The numbers on the arcs are the distances between the corresponding nodes.



i) Find the shortest route from node 1 to node 7. (2)

ii) Formulate the problem as a binary integer linear programming problem to get a route that is second best. (6)

6. a) Cars arrive at a toll gate according to a Poisson distribution with a mean of 90 cars per hour. The time for passing the toll gate is exponential with a mean of 38 seconds. Drivers complain of long waiting line and the authorities are willing to reduce the average passing time to 30 seconds by installing a automatic toll collecting devices provided two conditions are satisfied: i) the average number of waiting cars exceed 5 units and ii) the percentage of gate idle time with the new device installed does not exceed 10%. Can the device be justified? (4+4+2=10)

b) The probability P_n of n customers in the system $(M/M/1)(GD/5/\infty)$ are given in the table below. The arrival rate is 5 customers per hour.

n	0	1	2	3	4	5
P_n	0.399	0.249	0.156	0.097	0.061	0.038

The service rate is 8 per hour.

i) Compute the probability that an existing customer will be able to enter the system.

ii) Compute the probability at which the arriving customers will not be able to enter the system.

iii) Compute the expected number in the system.

iv) Compute the average waiting time in the queue.

(2+5+2+5=14)

Reliability

7. a) Consider the Weibull distribution with reliability function

$$R(t) = e^{-\left(\frac{t}{\theta}\right)^{\beta}},\ t \geq 0,\ \theta > 0,\ \beta > 0.$$

i) Discuss different types of failure rate of this distribution. (6)

ii) Find the mean and variance time to failure. (6+6=12)

b) Show the relationships between the failure density, hazard rate and reliability assuming that the hazard rate is a constant. (6)

8. a) Consider two simple non-repairable systems: i) a 2-out-of-3 system with identical components, and ii) a 3-out-of-4 system with identical components. All the components are assumed to be independent with constant failure rate λ. Derive the reliability functions and MTTF of both systems. Which system has higher reliability? (8)

b) Consider a system comprised of two serial components A and B having same reliability. Compute system reliability for both low-level and high-level redundancy for system with two parallel components. Which type of redundancy will give more reliable system? (8)

c) A component has a constant failure rate, λ, and a constant repair rate, r. Assume that the component is in one of the two possible states: operating or under repair. Find the probability that the component is operating at time t. Also, find the probability that the component is in operating condition during a time interval $[t_1\ t_2]$. (8)

Statistical Quality Control

9. Consider a drilling process in which the drill bit wears out with time. Assume that the length of the drill bit follows normal distribution. Assume that after using the drill bit for t hours, the mean of the distribution is $200 - \mu t$, where μ is a positive real number, and standard deviation 1 (the units of length here are in microns and for simplicity we are assuming that the variance is constant).

    (a) Describe the process here in statistical terms. (4)

    (b) Is the process in question a stable process? (4)

    (c) Name the type of control chart used for the process in question. Describe a stable process associated with this process. What type of control chart would you use for this stable process? (3+8+5=16)

10. (a) Give an example of a multivariate process. (5)

    (b) Consider a bivariate process $(x_1,\ x_2)$ that has bivariate normal distribution. Explain its parameters and describe how you estimate them. (10)

    (c) Is it necessary to assume that x_1 and x_2 are independent? What type of control is needed when they are independent? (2+2=4)

(d) Sketch the graphs of process region and control region for this process. (5)

Quality Management

11. a) What is policy deployment? How is it implemented and what are the benefits of it? (4+4+2=10)

    b) What is the advantage of using Lean Six Sigma instead of Lean or Six Sigma separately? (6)

    c) Describe a process with which you are familiar. List few causes that contribute to the common cause variation, and cite some examples of special cause variation in the process you have described. (2+6=8)

12. a) How will you build a house of quality in Quality Function Deployment (QFD)? (4)

    b) What are the definitions of quality provided by Juran and Crosby respectively? (2+2=4)

    c) Differentiate between internal failure costs and external failure costs. Explain four types of internal failure costs. (4+4=8)

    d) What is the difference between defect and defective of a product? Provide examples for both of them. (2+2=4)

    e) Explain conjoint analysis. How is it helpful in identifying the customer needs? (2+2=4)

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