Group A

1. (a) Find the values of A and B so that function f defined by

$$f(x) = \begin{cases} 2x^2 & \text{for } x \leq 2 \\ Ax + B & \text{for } x > 2 \end{cases}$$

is differentiable at $x = 2$.

(b) Let $(1+\sqrt{2})^n = A_n + B_n\sqrt{2}$ with A_n and B_n rational numbers.

i. Express $(1-\sqrt{2})^n$ in terms of A_n and B_n.

ii. Compute $\lim_{n\to\infty} \frac{A_n}{B_n}$

(c) Let A be an $n \times n$ real skew-symmetric matrix (that is $A^T = -A$). Prove that the matrix A is singular if n is odd. [9+(5+5)+5=24]

2. (a) We have n urns, each of which contains a white balls and b black balls. A ball is drawn from the first urn and put into the second, then a ball is drawn from the second and put into the third and so on. Then a ball is drawn from the last urn. Obtain the probability that this ball from the last urn is white.

(b) X is a standard normal variate, and Y is defined as

$$Y = \begin{cases} X & \text{if } -1 \leq X \leq 1 \\ -X & \text{otherwise} \end{cases}$$

Prove that the distribution of Y is also standard normal.

(c) A random variable has X has density $f_1(x)$ with probability p or a density $f_2(x)$ with probability $1-p$ with corresponding means and variances μ_1, μ_2 and σ_1^2, σ_2^2, respectively. Define $I = 1$ if X has density $f_1(.)$ and $I = 0$ if X has density $f_2(.)$. Derive the expressions for mean and variance of X in terms of p, μ_1, μ_2 and σ_1^2, σ_2^2. [8+8+(3+5)=24]

3. (a) Consider $x_1, \cdots, x_n$ independent observations from the common $\mathcal{N}(\mu, \sigma^2)$ distribution. Derive the maximum likelihood estimator of σ^2. Prove that it is biased. Hence obtain an unbiased estimator for σ^2.

(b) Consider the following data on men classified by smoking habit and mortality in six years.

→ / ↓	Non-smokers	Smokers
Dead	n_{11}	n_{12}
Alive	n_{21}	n_{22}

Suggest an appropriate test for equality of mortality in six years among smokers and non-smokers based on the above data, indicating the null and a suitable alternative hypothesis. Explain your notation clearly. [(6+6+2)+10=24]

Group B

1. (a) Consider the problem: maximize $c^{\mathrm{T}}x$ subject to $Ax = b$, where c and x are n-vectors, A is $m \times n$ matrix, b is m-vector and x is the vector of unknown decision variables. Show that the problem is either infeasible, or unbounded, or all its feasible solutions are optimal.

   (b) A carpenter, plumber and an engineer are available to perform the four given tasks. Each person can perform only one task in the alloted time, i.e., three of the four tasks can be completed. The objective is to find an assignment of the tasks to the persons that minimizes the total inefficiency (%). The inefficiency-matrix for person i assigned to task j is as follows for $i = 1, 2, 3$ and $j = 1, 2, 3, 4$.

Tasks →	Soldering	Framing	Drafting	Wiring
Carpenter	2	6	4	4
Plumber	3	4	4	3
Engineer	2	5	6	5

   (i) Formulate the above as an optimization problem to find which person should be assigned which job, and which job remains untouched.

   (ii) Solve the formulated problem. [10+(7+7)=24]

2. (a) i. Show that exactly one of the following system has a solution:
   System 1: $Ay \leq c, \quad y \geq 0$
   System 2: $x^T A \geq 0, \quad c^T x < 0, x \geq 0$

   ii. Let a pair of primal and dual problems be as follows:
   Primal : Minimize $c^T x$, subject to $Ax = b, \quad x \geq 0$
   Dual : Maximize $b^T y$, subject to $A^T y \leq c$, y-unrestricted
   Show that the above Primal problem is infeasible iff the Dual problem is unbounded with c replaced by 0 in the right hand side of the constraints of the Dual problem.

   (b) Let S be a convex set in $\mathbb{R}^n$, A be an $m \times n$ matrix. Show that $AS = \{y : y = Ax, \quad x \in S\}$ is convex.

   (c) Let S be a closed convex set in $\mathbb{R}^n$ and let $x \in S$. Suppose that d is a non-zero vector in $\mathbb{R}^n$ and $x + \lambda d \in S$ for all $\lambda \geq 0$. Show that d is a direction of S. [(6+6)+4+8=24]

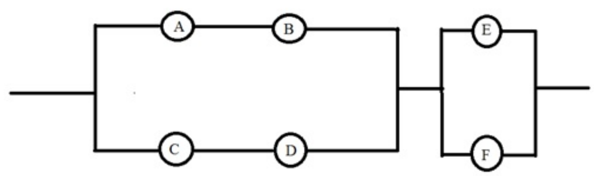
3. (a) A component has a reliability function given by

$$R(t) = 1 - \frac{t^2}{a^2}, \text{ for } 0 \leq t \leq a,$$

where a is a parameter of the distribution representing the component's maximum life.

i. Find the hazard rate.
ii. Find the mean time to failure.
iii. Find the median life.

(b) For the following network, derive an expression for the system reliability in terms of the component reliabilities. Assume that each component has a reliability R.



Identify the minimal path sets and the minimal cut sets. Write the structure function using either the minimal path sets or the minimal cut sets.

(c) An aircraft requires three out of its four main engines to operate in order to complete its journey of ten units of time successfully. If each engine has a constant failure rate λ, determine the reliability of completing the journey successfully.[(3+2+2)+(4+2+2+3)+6=24]

4. (a) A total of n independent light bulbs each having life distribution $F(t) = 1 - \exp[-(\frac{t}{\lambda})]$, $\lambda > 0$, are put to test. The observations on failure times are subject to random censoring from the right. Find an explicit expression for the maximum likelihood estimator of the median life of the light bulbs. Explain the data and the notation clearly.

(b) i. Consider a series system with two independent components having exponential life distributions with common hazard rate λ. When one of the two components fails, a third stand-by component, independent of the original two but with the same constant hazard rate λ, automatically replaces the failed component immediately without causing failure of the system. What is the mean life time of this whole system?

ii. We have n such systems, as in 4(b)i, under observations, so that we observe both the replacement time (r_i) and the failure time (t_i) for each of the n systems. Find the maximum likelihood estimate of λ. [10+6+8=24]

5. (a) A particular quality characteristic of a manufacturing process follows a Normal distribution with mean μ and standard deviation σ. The upper and lower specification limits for the quality characteristic are symmetrical and specified by U and L respectively, along with a target value T. The standard notations for capability indices C_p, C_{pu}, C_{pl} and C_{pk} and non-conformance indices NC_L, NC_U and NC_{Total} are defined. Note that NC_L is proportion non-conformance on the lower side and NC_U is proportion non-conformance on the upper side.

   i. Find the relationship between (1)NC_U and C_{pu}, and (2) NC_L and C_{pl}.

   ii. Write NC_{Total} in terms of C_p and C_{pk}.

   (b) Let X be a quality characteristics which follow Normal distribution with mean μ and standard deviation 1. The target value of this quality characteristics is zero(0) and has symmetrical specification limits given by L and U. The process capability indices of this characteristics are $C_p = 1.5$ and $C_{pu} = 1$. Compute the values of μ, L & U and expected non-conformance for the quality characteristics.[$P(z < 3) = 0.99865$]

   (c) Let L and U be the lower and upper specification limits of a process A. Let μ_A and σ_A be the process average and standard deviation, respectively. Assume that $\mu_A = T$, where $T = \frac{L+U}{2}$, making $C_p(A) = C_{pk}(A)$. Now consider a process B with the same specification limits of the process A but with $\mu_B = T + \epsilon, \quad \epsilon > 0$. Clearly, the process B is not centred. In particular, if $\mu_B = T + \frac{U-L}{2^k}$ and $\sigma_B = (1 - \frac{1}{2^{k-1}})\sigma_A$, where $k > 2$ is a positive integer, what can you say about C_{pk} of the process B?[(4+4)+6+10=24]

6. (a) Compare the philosophy of Dr.Deming with that of Dr.Phil Crosby.

   (b) State the components of quality cost and explain how each component of quality cost impact the profitability of the business.

   (c) How Six Sigma and ISO 9001 Quality Management system can be integrated in an organisation.

   (d) Explain the usefulness of control chart during Six Sigma implementation.

   (e) Write a note on quality 4.0 not exceeding ten sentences.

[6+6+4+4+4=24]