Group A

Mathematics

1. (a) Find the trace of the matrix $\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}^{20}$.

   (b) Let $W = \{p(B) : p$ is a polynomial with real coefficient$\}$, where

   $$B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

   Then evaluate the dimension of the vector space W.

   (c) Let P_n denotes the space of all polynomials $p(x)$ with coefficients in $\mathcal{R}$, such that the degree of $p(x) \leq n$, for a positive integer n and B be the standard basis of P_n. Let us consider a linear transformation T from P_3 to P_4. The transformation T is given by

   $$T(P(x)) = x^2 p'(x) + \int_0^x p(t)dt.$$

   If A is a (5×4) matrix of T, generated under this transformation, construct A.

   (d) Suppose $A^{3\times 3}$ is a symmetric matrix such that

   $$\begin{pmatrix} x & y & 1 \end{pmatrix} A \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = xy - 1.$$

   Let p be the number of positive eigen values of A and $q = rank(A) - p$. Find the value of q.

[5+6+7+6=24]

2. (a) $A^2x + 2Ax + \lambda x = 0, x \in \mathcal{R}^n$, $A^{n\times n}$ is a real matrix. Find the set of $\lambda \in \mathcal{R}$, for which the set of simultaneous equations have unique solutions, for all symmetric A.

(b) Let us consider the following two equations in x, y, z, where $a \in \mathcal{R}$.

$$2x + y + z = a$$

$$x + 2y + z = 0.$$

What is the dimension of the solution space in this context? Describe all the solutions in terms of a.

(c) Find $\lim_{n\to\infty}(1 + \frac{\sin n}{n \log n})^{\sqrt{n}}$.

(d) $f(x) = \begin{cases} \exp(-\frac{1}{x^2}), & -\infty < x < \infty, x \neq 0 \\ 0, & x = 0 \end{cases}$

Is the function $f(x)$ continuous at $x = 0$? If so, is it differentiable?

[8+(3+5)+4+4=24]

Probability & ***Statistics***

3. (a) Suppose X be a standard normal variable. Show that $P(x^3 - 2x^2 - x + 2 > 0) = 2\Phi(1) - \Phi(2)$.

(b) Let $g(x)$ be a function with $-\infty < E(g(X)) < \infty$ and $-\infty < g(-1) < \infty$. If $X \sim Poisson(\lambda)$, show that $E(\lambda g(X)) = E(Xg(X-1))$. Hence find the variance of X using the above relationship of expectation.

(c) Suppose X has the density $f(x|\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}}, x > 0$, where $\theta > 0$ is unknown. Define $Y = k$, if $k \leq X < k+1, k = 0, 1, 2, \ldots$. Show that the distribution of Y is geometric with success probability $\left(1 - e^{-\frac{1}{\theta}}\right)$. Let us consider an estimator $T_n^\alpha = [\frac{1}{n}\Sigma x_i^\alpha]^{\frac{1}{\alpha}}$ for θ, where $\alpha \in [-1, 1]$. Show that the limiting value of T_n^α converges to sample geometric mean when $\alpha \to 0$.

[6+8+(5+5)=24]

4. (a) Suppose $X_1, X_2 \dots X_n (n > 1)$ be a random sample from $U(2\theta - 1, 2\theta + 1)$, where $\theta \in \mathcal{R}$. Let us define, $Y_1 = Min(X_1, X_2 \dots X_n)$ and $Y_n = Max(X_1, X_2, \dots X_n)$. Construct a suitable range based on Y_1 and Y_n so that every statistic lies within the range can be regarded as a MLE of θ.

(b) Find the MLE of θ based on only two observations -1 and 1, from $N(\theta, \theta)$.

(c) Suppose X follows $N(0, 1)$ and $(Y|X = x)$ follows $N(\alpha x, 1)$, $0 < \alpha < 1$. Express the coefficient of determination in terms of α.

(d) Let X be a random variable of continuous type with probability density function

$$f(x|\theta) = \begin{cases} \frac{\theta}{x}\left(\frac{3}{\theta}\right), x > 3, \theta > 0 \\ 0, \text{otherwise.} \end{cases}$$

Based on the single observation X, find the most powerful test of size $\alpha = 0.1$, for testing $H_0 : \theta = 1$ against $H_1 : \theta = 2$.

(e) $\mathbf{X} = \begin{pmatrix} X_1 & X_2 & X_3 & X_4 \end{pmatrix}' \sim N_4(0, \Sigma)$, where

$\Sigma = \begin{pmatrix} 1 & \rho & \rho & \rho \\ \rho & 1 & \rho & \rho \\ \rho & \rho & 1 & \rho \\ \rho & \rho & \rho & 1 \end{pmatrix}$ is a positive definite matrix. Find the distribution of $\{(X_1 - X_3)^2 + (X_2 - X_4)^2\}\frac{1}{2(1-\rho)}$.

[5+4+6+5+4=24]

Group B

Operations Research

5. (a) Let $h(x, y) = \max\{e^x - 1, y^2\}$ where $(x, y) \in \mathcal{R}^2$.

(i) Is $h(x, y)$ a convex function over $\mathcal{R}^2$?

(ii) Find the minimum value of $h(x, y)$, when $(x, y) \in \mathcal{R}^2$.

(iii) Suppose $(x^\circ, y^\circ) \in \mathcal{R}^2$ with $h(x^\circ, y^\circ) \leq h(x, y)$ for every $(x, y) \in \mathcal{R}^2$. Is (x°, y°) unique? Give reasons.

(b) Maximize $2x_1 + 4x_2 + x_3 + x_4$
subject to :

$$\begin{array}{ccccccc} x_1 & + & 3x_2 & + & x_4 & \leq & 4 \\ 2x_1 & + & x_2 & & & \leq & 3 \\ x_2 & + & 4x_3 & + & x_4 & \leq & 3 \\ x_1, & x_2, & x_3, & x_4, & & \geq & 0 \end{array}$$

(i) Write down the dual of the given problem.

(ii) Examine whether (1, 1, 1/2, 0) and (11/20, 9/20, 1/4) are optimal solutions for the respective problems.

(c) Solve the following LPP

Minimize $2x_1 + 5x_2 + 4x_3 + x_4$
subject to :

$$\begin{array}{ccccc} x_1 & + & x_2 & = & 7 \\ x_1 & + & x_3 & = & 8 \\ x_3 & + & x_4 & = & 7 \\ x_2 & + & x_4 & = & 6, \end{array}$$

$x_1, x_2, x_3, x_4 \geq 0.$

[(3×4)+(2+6)+4= 24]

6. (a) Let $C = \{x \in \mathcal{R}^n : Ax \leq 0\}$, where A is an $m \times n$ matrix with real entries. Suppose C has an extreme point. Is it unique? Give reasons for your answer.

(b) (i) Each of the two players A and B toss a coin once. If the outcomes match, the player A pays B an amount \$a. If the outcomes do not match, the player B pays A an amount \$b. Formulate this problem into a zero sum matrix game and solve it.

(ii) Consider the two person zero sum game with payoff matrix for A is given by :

		B	
		I	II
A	I	a	7
	II	-3	a

Find the range of a, for which the game is strictly deterministic.

(c) Minimize $X^2 + Y^2$
subject to :

$$\begin{aligned} X + 2Y &\le 5 \\ X - Y &\ge 2 \\ Y &\ge 1 \end{aligned}$$

X is unrestricted in sign.
Find the solution of the above optimization problem.

(d) Let A be a $m \times n$ matrix with real entries, $b \in R^m$. Using Farkas lemma prove that exactly one of the following systems has a solution

(i) System I : there exists $x \in \mathcal{R}^n, Ax \le b$

(ii) System II : there exists
$y \in \mathcal{R}^m, y \ge 0, y^t A = 0$ and $y^t b < 0$.

[4+(4+4)+4+(4+4) = 24]

Reliability

7. (a) Consider a parallel system with two independent components. The lifetimes of each component follow Weibull distribution with scale parameter θ and shape parameter β. Find the reliability function and failure rate of the system.

(b) Consider a coherent system with five independent components 1, 2, 3, 4 and 5 in which the minimal path sets are $1, 3, 5,\ 2, 3, 5$ and $2, 4, 5$.

(i) Write down the structure function of the system.

(ii) Write down the minimal cut sets of the system.

(iii) Assuming that each component has a constant failure rate λ, find the reliability of the system at time $t > 0$.

(iv) Find the lower bound and upper bound of reliability of the system at time t > 0.

[(4+4)+(4×4)=24]

8. (a) One hundred mobile phones of a particular brand are observed at 1-year intervals with the following number of failures: $20, 40, 30, 10$; without replacement.

   (i) Estimate reliability function $R(t)$ and hazard rate $\lambda(t)$ for $t = 1, 2$ year.

   (ii) Determine the sample mean time to failure.

(b) Suppose n number of identical units were placed on a life test and $r \leq n$ be the number of failures with $t(1), t(2), t(3), \ldots,$ $t(r)$ as their ordered failure times. Assume that the lifetime distribution to be exponential with parameter λ. Find the maximum likelihood estimate of the mean time to failure of the units.

(c) Suppose you have to ensure a system reliability of 99.99 percent with several components, each with reliability of 90 percent. Find the number of redundancy.

(d) Consider a system comprised of two serial components A and B, having the same reliability. Two similar components can be added into the system either with component-level redundancy or with system-level redundancy. Which type of redundancy will give more reliable system?

[(6+2)+8+4+4 =24]

Statistical Quality Control

9. (a) What type of conclusions about a process can be made from an individual and moving range (IMR) control chart that cannot be made from histogram?

(b) What assumptions must be made before a Multivariate Shewhart Control chart can be used for controlling the multivariate process mean?

(c) Explain the difference between process capability index and process performance index.

(d) Suppose there is strong evidence that a process characteristic has a non-normal distribution. What are the possible approaches for computing process capability index?

[[**6+6+6+6 =24**]

10. (a) Describe the working procedure of single and double sampling plans for attributes. Discuss the relative advantages and disadvantages of a double sampling plan with respect to a single sampling plan.

(b) Discuss Producers risk and Consumers risk related to sampling inspection.

(c) A fraction non-conformance chart is to be designed with average fraction non-conformance as 0.15 and with three sigma limits. What should be the sample size in order to get a positive lower control limit?

(d) What is out of control action plan (or reaction plan)? How it will be helpful in process monitoring?

[**6+6+6+6 =24**]

Quality Management

11. (a) Mention different versions of ISO 9000 series of Quality Management System Standards that has been in use from 1987 onwards.

(b) In all the above versions, management review is one of the important activities to be performed by the Management seeking certification by an organization. Describe what need to be done under management review as per the ISO 9001: 2015 standards requirement.

(c) Explain why it is important to separate sources of variability into special (or assignable) causes and common (or chance) causes?

(d) What is the difference between DMAIC and DMADV approaches in Six Sigma? Explain with examples.

[4+8+6+6=24]

12. (a) Seven tools for quality management and new seven tools for quality management has been the back bone for managing an organisations quality initiatives. Describe briefly any 12 of these quality management tools explaining how it helps in improving the process.

(b) What is the idea behind using DPMO metric in Six Sigma? What are its limitations?

[12+12=24]