Group A

Mathematics

1. (a) Let $S = \{(x, y, z) \in R^3 : (x-y)(y-z) = 0\}$. Find the dimension of span of S.

   (b) Let
$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ a & 0 & 1 \end{bmatrix}.$$
   For what values of a, A is invertible? If $a = 2$, find A^{-1}.

   (c) Solve the differential equation
$$\frac{1 + \frac{dy}{dx}}{x - y} = \frac{1 - \frac{dy}{dx}}{x + y}.$$
   What curves does it give?

[8+(4+3)+(7+2)=24]

2. (a) Suppose $\{x_n\}$ is a real-valued sequence such that $x_n^2 - 2 \to 0$. Let $\{x_{n_i}\}$ be a sub-sequence of $\{x_n\}$. Does $\{x_{n_i}\}$ always have a limit? If $\{x_{n_i}\}$ converges, find all possible values of limit of $\{x_{n_i}\}$.

   (b) Let $A_{6\times6}$ and $B_{6\times6}$ be two real matrices with rank$(A) = 4$ and $A + B$ is non-singular. What are the possible ranks of B?

   (c) A function f is defined for $x \in R$ as follows
$$f(x) = \begin{cases} x^x, & \text{for } x > 0 \\ 0, & \text{for } x \leq 0. \end{cases}$$
   Find $\lim_{x\to0+} f(x)$.

   (d) Show that $\frac{x}{1+x} < \log_e(1 + x) < x$, if $x > 0$.

[6+4+6+8 = 24]

Probability & Statistics

3. (a) Let $X_1, \ldots, X_n$ be a random sample from $N(\theta, \sigma^2)$. Find the maximum likelihood estimator of σ^2 if θ is known. Is it UMVUE?-Justify. Find a $100\times(1-\alpha)\%$ confidence interval for σ^2 if θ is known.

   (b) Let $Z_1, \ldots, Z_n$ be independent and identically distributed with cumulativfe distribution function $F(z) = 1 - \exp(-\lambda z)$, $z > 0, \lambda > 0$. Define $U = \min\limits_{1\le i\le n} Z_i$ and $V = \max\limits_{1\le i\le n} Z_i$. Find the joint distribution of U and V. Show that U and $V-U$ are independently distributed.

[(4+4+4)+(8+4)=24]

4. (a) Let X_1, X_2 and X_3 be independent and identically distributed random variables each having probability density function $f(x) = \lambda e^{-\lambda x},\ x > 0,\ \lambda > 0$. Find $P(X_1 > (X_2 + X_3)/2)$.

   (b) Let $\{X_n, n = 0, 1, 2, \ldots,\}$ be a Markov chain with state space $S = \{0, 1, 2\}$ and transition probability matrix

$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \alpha & 0 & 1-\alpha \\ \beta & 0 & 1-\beta \end{bmatrix}.$$

   (i) Find the values of α and β for which the Markov chain is irreducible.

   (ii) Suppose that $\alpha = \beta = \frac{1}{2}$. Find the stationary distribution of the Markov chain, if it exists.

   (c) Let $X_{11}, X_{12}, \ldots, X_{1n_1}$ and $X_{21}, X_{22}, \ldots, X_{2n_2}$ be independent random samples from exponential distributions with respective means θ_1 and θ_2. Derive a consistent level α test for $H_0 : \theta_1 = \theta_2$ versus $H_1 : \theta_1 \neq \theta_2$.

[8+(4+4)+8 = 24]

Group B

Operations Research

5. (a) Consider the following linear programming problem

$$\text{Maximize } z = c^t x$$
$$\text{subject to } Ax \leq b$$
$$x \geq 0, b \geq 0,$$

where $x, c \in R^n$, $b \in R^m$, and A is an $m \times n$ matrix. Let y^* denote the optimal solution for the dual problem. Suppose the right hand side vector b is changed to $\overline{b}$ in the primal problem, and let $\overline{x}$ denote the optimal solution to the new primal. Show that

$$c^t \overline{x} \leq y^* \overline{b}.$$

(b) Derive an optimal solution to the following

$$\text{Maximize } f(x) = 8x_1 - x_1^2 + 4x_2 - x_2^2$$
$$\text{subject to } x_1 + x_2 \leq 2$$
$$x_1, x_2 \geq 0.$$

(c) Consider the inventory problem in which stock is replenished instantaneously at a uniform rate a. Let $a > D$, where D is the consumption rate. The setup cost is K per order and the holding cost is h per unit time. If y is the order size and no shortage is allowed,

(i) Show that the maximum inventory level is $y\left(1 - \frac{D}{a}\right)$.

(ii) What is the total cost per unit time for a given y?

(iii) What is the Economic Order Quantity?

[8+8+(2+3+3)= 24]

6. (a) A petrol bunk with only one pump employs the following policy. If a customer has to wait, the price is Rs 75/ litre. If he does not have to wait, the price is Rs 90/ litre. Customer arrive at the station according to a Poisson Process with a mean rate of 12 per hour. Service time at the pump have an exponential distribution with a mean of 10/3 minutes. Arriving customers wait until they eventually buy petrol. Determine the expected price of petrol.

(b) In the two player, two finger Morra game, each player shows one or two fingers and simultaneously guesses the number of fingers the opponent will show. The player making the correct guess wins an amount equal to the total number of fingers shown. Otherwise the game is draw. Setup the problem as a Two-person zero-sum game and answer the following.

(i) Determine the pay-off matrix.

(ii) Formulate the above game problem as a linear programming problem.

(iii) Comment about the value of the game and comment on the optimal strategies of the two players.

(c) Rashtriya Bank has one outside drive-up teller. It takes the teller an average of 4 minutes to serve a bank customer. Customers arrive at the drive-up window at a rate of 12 per hour. The bank operations officer is currently analyzing the possibility of adding a second drive-up window, at an annual cost of Rs.200,000. It is assumed that arriving customers would be equally divided between both windows. The operations officer estimates that each minute's reduction in customer waiting time would increase the bank's revenue by Rs 20,000 annually. Should the second drive-up window be installed? Justify your answer.

[8+(3+3+2)+8=24]

Reliability

7. (a) The following observations are failure times in days for a sample of 6 electronic components in a life test.

$$2,\ 3+,\ 4+,\ 6,\ 6+,\ 7,$$

where + indicates right censored observations.
Suppose that the lifetimes of the components are independent and identically distributed with probability density function

$$f(t) = \frac{t}{\theta^2} \exp\left(-\frac{t^2}{2\theta^2}\right),\ t \geq 0,\ \theta > 0.$$

Find the maximum likelihood estimate (MLE) of reliability of a component at 5 days. Find the MLE of median lifetime of a component.

(b) A device is replaced by a spare in the event of its failure or having reached the age of T_0, whichever is earlier. Suppose the lifetime of the device has constant hazard rate λ. Find the expected replacement time of the device.

(c) A repairable system has a constant failure rate λ and a constant repair rate β. Assume that the system is either in operating condition or in under repair condition. Find the probability that the system is operating at time t.

[(7+3)+7+7=24]

8. (a) Consider a coherent system with four components 1, 2, 3 and 4, in which the minimal path sets are $\{1,2\}$, $\{3,4\}$ and $\{1,4\}$.

   (i) Write down the structure function of the system.

   (ii) Suppose that the component lifetimes are independent and identically distributed each with constant hazard rate λ. Find the reliability function of the system.

   (b) Consider a series system with n components. The lifetimes of the system are independent and identically distributed each having reliability function

$$R(t) = e^{-\frac{t^\beta}{\alpha}},\ t \geq 0,\ \alpha > 0,\ \beta > 0.$$

   (i) Find the hazard rate of the system lifetime.

   (ii) Check whether the system lifetime distribution is IFR or DFR.

   (iii) Find the mean time to failure of the system.

[(4+4)+ (8+4+4) =24]

Statistical Quality Control

9. (a) What are the various reasons of separating out the variations due to the chance cause and special cause in Statistical Process Control?

   (b) Why a single multivariate control chart is recommended rather than using multiple univariate control chats in multivariate process monitoring with correlated characteristics?

   (c) Recommend appropriate control charting method for a process with very low non-conformance rate (may be less than 1000 per million).

(d) What is the use of 'rational subgrouping' in Shewhart control charts?

[6+6+6+6 =24]

10. (a) Two machines in a manufacturing unit produce same products. Machine-wise Cp and Cpk values are as follows. Machine 1: Cp = 1.4, Cpk = Cpu = 0.85. Machine 2: Cp = 1.4, Cpk = Cpl = - 0.85. where, Cp: Potential process capability index; Cpk: Process performance index; Cpu: Process performance index with respect to USL; Cpl: Process performance index with respect to LSL. Sketch the distribution of the critical quality characteristic of the product with respect to its specification limits (USL and LSL) based on the indices Cp and Cpk. Comment about the process centering and variability of these two processes.

(b) Why is the np chart not appropriate with variable sample size?

(c) A control chart for the fraction nonconforming is to be established using a center line of $p = 0.10$. What sample size is required if we wish to detect a shift in the process fraction nonconforming to 0.20 with probability 0.50?

(d) Consider a single sampling plan with rectification scheme where all defective units are removed but not replaced with good ones. Find the AOQ.

[6+4+6+8 =24]

Quality Management

11. (a) Explain the impact of cost of quality on organizational performance.

    (b) What is top-down approach in project selection of six sigma methodology? Explain.

    (c) Explain the link between continuous improvement and PDCA cycle.

    (d) Explain the role of conjoint analysis in determining the customer perceptions.

[6+6+6+6=24]

12. (a) Discuss the preachings of any three Quality Gurus in *Six Sigma*.

    (b) What is the unique feature of *Six Sigma* which is not discussed by the above Quality Gurus.

    (c) Discuss any two merits and demerits of the feature mentioned in the previous part.

[9+5+(5+5)=24]