Group A

Mathematics

1. (a) For a symmetric and non-negative definite matrix $A_{n\times n}$, show that the matrix $B = I - A - A^2 + A^3$ is always non-negative definite, where I is identity matrix.

   (b) For a symmetric matrix A, let $A^n \to B$, as $n \to \infty$ (entry-wise convergence) with real finite matrix B. Show that trace(B) is an integer.

   (c) Consider a 4×4 square. Find the number of ways the sixteen entries of it can be filled up with 0 or 1, so that each row sum and column sum gives odd numbers.

$[8 + 8 + 8 = 24]$

Probability & Statistics

2. (a) Let (X, Y) have uniform distribution on the square generated by the points $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$. Calculate the following probabilities.

   (i) $P(X \leq 0|Y \leq 0)$.

   (ii) $P(X \leq \frac{1}{3}|X + Y \geq \frac{1}{2})$.

   (b) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed continuous $uniform(0, 1)$. Let $U = \min(X_1, X_2, \ldots, X_n)$ and $V = \max(X_1, X_2, \ldots, X_n)$. Find the joint distribution of U and V.

   (c) Suppose X is an observable random variable with probability density function $f(x)$, $x \in R$. Consider two functions f_0 and f_1 defined as follows

$$f_0(x) = \begin{cases} \frac{3}{64}x^2, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise} \end{cases}$$

and

$$f_1(x) = \begin{cases} \frac{3}{16}\sqrt{x}, & \text{if } 0 < x < 4 \\ 0, & \text{otherwise.} \end{cases}$$

Find the most powerful level α test for testing $H_0 : f(x) = f_0(x)$ versus $H_1 : f(x) = f_1(x)$.

$[(3 + 5) + (8 + 8) = 24]$

3. (a) Let $X_1,\ X_2,\ X_3,\ Y_1,\ Y_2$ be independent and identically distributed $N(1, 5)$. Let $U = (X_1 + X_2 + X_3)/4$ and $V = (Y_1 + Y_2)/3$. Find $P(U > V)$ in terms of cumulative distribution function of $N(0, 1)$.

(b) Let x_1, x_2, x_3, x_4, x_5 be five independent observations from the probability density function

$$f(x) = \begin{cases} e^{\theta - x}, & \text{if } x > \theta \\ 0, & \text{otherwise.} \end{cases}$$

(i) Find the maximum likelihood estimate of θ.

(ii) Give a sufficient statistics of θ.

(iii) Give an unbiased estimate of θ.

(c) Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed $N(1, 2)$. Let $\bar{X}_n = \sum_{i=1}^{n} X_i/n$ and $S_n = \sum_{i=1}^{n}(X_i - \bar{X}_n)^2$. Let $Y_n = \sqrt{K + n}\left(\frac{\bar{X}_n - 1}{\sqrt{\frac{S_n}{n+L}}}\right)$, where K and L are fixed positive real numbers. Find the limit of Y_n (in distribution) as $n \to \infty$.

$[8 + (2 + 2 + 4) + 8 = 24]$

Group B

Operations Research

4\. (a) A company is producing a product which requires three parts at the final assembly stage. These three parts can be produced by two different departments as detailed below:

	Production rate (units per hour)			Cost (Rs. per hour)
	Part 1	Part 2	Part 3	
Department 1	7	6	9	2500
Department 2	6	11	5	1250

In a week, 1050 finished (assembled) products are needed but up to 1200 can be produced. Suppose Departments 1 and 2 have 100 and 110 working hours available in a week, respectively. There is a constraint in storage space and a maximum 200 unassembled parts (of all types) can be stored at the end of the week. Formulate the problem of minimizing the cost of producing the finished (assembled) products needed for the week as a linear programming problem.

(b) Consider the following linear programming problem:

$$\text{Maximize } z = \mathbf{c}^T\mathbf{x}$$
$$\text{subject to } A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0,$$

where $\mathbf{c}, \mathbf{x} \in R^n, \mathbf{b} \in R^m$ and A is an $m \times n$ matrix. Call this problem as P. Write down the dual (D) of this problem. Show that P is infeasible if and only if the homogeneous version of D (with the right hand side of D replaced by zeros) is unbounded.

$[14 + (3 + 7) = 24]$

5. (a) Consider the following problem:

$$\text{Minimize } x^2 + (xy - 1)^2$$
$$\text{subject to } \; x \geq 0, y \geq 0.$$

Find the minimum value of the problem. Is the minimum value attained? Justify your answer.

(b) Customers arrive to purchase hall tickets at an average rate of 3 per minute. It takes an average of 15 seconds to purchase a ticket. It takes exactly 1 minute to reach the correct seat after purchasing the ticket. If a customer arrives 2 minutes before the show starts,

(i) can he expect to be seated just before the start of the show? Justify your answer.

(ii) what is the probability that he will be seated at the start of the show? (Given $e^{-1} = 0.37$)

(c) Consider a two person zero-sum game. Suppose the pay-off table for Player 1 is given by

	Player 2		
Player 1	0	−2	2
	5	4	−3

Find the optimal strategy of Player 1.

$[(3 + 3) + (5 + 5) + 8 = 24]$

Reliability

6. (a) Consider a coherent system with five components 1, 2, 3, 4 and 5, in which the minimal path sets are $\{1, 3\}$, $\{2, 3\}$ and $\{4, 5\}$.

(i) Find the minimal cut sets of the system.

(ii) Write down the structure function of the system.

(b) Consider a system consists of 5 identical and independent components with each component having a constant failure rate λ. The system functions if and only if at least 3 components function.

   (i) Find the reliability function of the system.

   (ii) Find the mean time to failure of the system.

(c) Find the hazard rate of a system whose mean residual life at t is θ, a constant.

$[(4 + 3) + (6 + 6) + 5 = 24]$

7. (a) Consider a parallel system with n independent components. The system is subject to random stress Y following an exponential distribution with mean μ_y. The strengths of the components $X_1, X_2, \ldots, X_n$ are independent and identically distributed random variables, each having exponential distribution with mean μ_x. The system fails when the stress exceeds strength. Assuming that the stress is independent of the strengths, find the reliability of the system.

   (b) Suppose 15 items are put on a life test at time zero. The test is stopped when 10 items fail. Suppose the total lifetime of the failed items is 150 days and the largest failure time is 40 days. Assuming that the lifetime follows exponential distribution with mean θ, find the maximum likelihood estimate of reliability at 40 days.

   (c) Suppose a repairable system is observed until n failures occur and the observed failure times are $t_1 < t_2 < \cdots < t_n$. Assume that the failure occurs according to a Poisson process with rate λ. Derive the maximum likelihood estimate of mean time between failure μ. Is it an unbiased estimator of μ? Justify.

$[7 + 5 + (8 + 4) = 24]$

Statistical Quality Control

8. (a) A consumer receives candles in lots of size 500 each from a supplier. To check the quality of a lot, the consumer draws a sample of size 20 and accepts the lot if the inspected sample contains at most one defective candle. Otherwise, he/she rejects the lot. Under this sampling plan, the probability of accepting a lot of quality $p = 0.02$ is 0.94. The management decides to screen all the rejected lots and replace all the defective candles by the non-defective ones. If the submitted lot quality is $p = 0.02$, calculate average outgoing quality (AOQ) and average total inspection (ATI).

   (b) Explain short run processes with an example. Describe a method for monitoring the short run processes.

   (c) Briefly describe procedures for assessing stability of a process when (i) sub-group size is one, and (ii) sub-group size is more than one.

   Suppose the upper and lower specification limits for the quality characteristic are specified as U and L, respectively. Describe procedures of estimating the process capability indices for the above two cases. Assume that the quality characteristic follows a normal distribution.

$[(4 + 4) + (4 + 4) + (4 + 4) = 24]$

9. (a) Define in-control and out-of-control average run length (ARL) in case of $\bar{X}$-chart. Derive the expressions for both the ARLs.

   (b) A product has k quality characteristics. Suppose each quality characteristic (continuous variable) is monitored separately using $\bar{X}$-charts.

(i) If the quality characteristics are independent, what is the probability that all the variables exceed their control limits simultaneously?

(ii) Describe an appropriate method for simultaneous monitoring of all the k quality characteristics when they are not statistically independent.

(c) A particular quality characteristic of a manufacturing process follows a normal distribution. The upper specification limit (USL) and lower specification limit (LSL) of the quality characteristic are symmetrical around a target value. A process capability study was conducted to evaluate the performance of the manufacturing process. The potential capability index C_p is estimated as 1.5 and the performance capability index with respect to LSL, C_{pl}, is estimated as 1.0.

(i) Find the amount of shift of the process mean compared to the target.

(ii) Find the estimate of expected fraction non-conformance from this process.

$[(4 + 6) + (2 + 6) + (3 + 3) = 24]$
