1. Suppose $\{a_n : n \geq 1\}$ and $\{b_n : n \geq 1\}$ are two sequences of positive real numbers such that $\limsup_{n\to\infty} \frac{1}{n} \log b_n = -\infty$. Show that

$$\liminf_{n\to\infty} \frac{\log(a_n + b_n)}{n} = \liminf_{n\to\infty} \frac{\log a_n}{n}.$$

[12]

2. Let $T = \{(x, y, z) \in \mathbb{R}^3 : 5x^2 + y^2 + z^2 + 4xy + 2xz = 0\}$. Show that T is a subspace of $\mathbb{R}^3$, and find a basis of T.

[12]

3. Consider the following matrix C of order $mn \times mn$:

$$C_{mn\times mn} = \begin{bmatrix} A & B & B & B & \dots & B \\ B & A & B & B & \dots & B \\ B & B & A & B & \dots & B \\ \vdots & \vdots & \vdots & \vdots & \dots & \vdots \\ B & B & B & B & \dots & A \end{bmatrix},$$

where $A_{m\times m} = aI_{m\times m} + bJ_{m\times m}$, $B_{m\times m} = dJ_{m\times m}$. $I_{m\times m}$ and $J_{m\times m}$ are the identity matrix and the matrix of all ones, respectively, a, b, d are nonzero integers, and $b \neq d$. Obtain the eigenvalues of the matrix C along with their algebraic multiplicities.

[12]

4. Let X_1, X_2, X_3, X_4 be an independent and identically distributed sample from an exponential distribution with mean $\lambda > 0$. Find the correlation coefficient between $X_{(2)}$ and $X_{(3)}$, where $X_{(r)}$ is the r-th order statistic.

[12]

5. Routes A and B are the only two escape routes from a state prison. Prison records show that 40% of the prisoners who tried to escape used route A. These records also show that 80% of those who tried to escape via A, and 70% of those who tried to escape via B were captured.

   (a) What is the expected number of attempts that a prisoner needs to make to successfully escape from the prison?

   (b) Given that two prisoners have independently and successfully escaped from the prison, what is the probability that they have used the same route to escape?

$[4 + 8 = 12]$

6. Based on 58 pairs of (x, y), it was observed that the sample means and the slope of the least squares regression line of y on x are:

$$\overline{x} = 16, \quad \overline{y} = 14, \quad b_{yx} = 1.2.$$

   (a) Subsequently, it was found that a pair of (x, y) was not recorded, and it was $(16, 14)$. Obtain the least squares regression line of y on x based on all 59 pairs of data.

   (b) Now we wish to include another (x, y) pair in the dataset, and it is $(16, 12)$. Find the slope of the least squares regression line of y on x based on all 60 pairs of data.

$[6 + 6 = 12]$

7. Let Y be a random variable with finite expectation, and m be a median of Y , i.e., $P(Y \leq m) \geq 1/2$ and $P(Y \geq m) \geq 1/2$. Show that, for any real numbers a and b such that $m \leq a \leq b$ or $m \geq a \geq b$, $E|Y - a| \leq E|Y - b|$.

[12]

8. Suppose $X_1, X_2, \ldots, X_n$ is a random sample from Uniform $\left[\theta-1, \theta+2\right]$, $\theta \in \mathbb{R}$ being unknown. Show that there exists a maximum likelihood estimator of θ in the form $\left(aX_{(1)} + bX_{(n)} + c\right)$ where a, b, and c are positive constants, and $X_{(r)}$ is the r-th order statistic.

[12]

9. Suppose X_1, X_2 and X_3 are positive valued i.i.d. (independent and identically distributed) non-degenerate random variables with finite variances.

   (a) Define $Y = X_1X_2$ and $Z = X_2X_3$. Prove that $0 < \rho < \frac{1}{2}$, where ρ is the correlation coefficient between Y and Z.

   (b) Prove that $E\left[\dfrac{X_1 + X_2}{\sqrt{{X_1}^2 + {X_2}^2 + {X_3}^2}}\right] < \dfrac{2}{\sqrt{3}}$.

[$6 + 6 = 12$]

10. Consider the linear model $Y = X_1\beta_1 + X_2\beta_2 + \epsilon$ where the matrix $[X_1 : X_2]$ has full column rank, and the number of sample points is larger than the number of regressors. Suppose $\widehat{\beta_2}$ is the least squares estimator of β_2 in the above model. Alternatively, one analyses the linear model $Y = X_1\beta_1 + \epsilon_1$ such that the vector of least squares residuals is Z. If the least squares estimator of β_2 based on the linear model $Z = X_2\beta_2 + \epsilon_2$ be denoted as $\widehat{\beta_2^*}$, show that

$$< X_2\widehat{\beta_2^*}, X_2\widehat{\beta_2} > \;\; \leq \;\; \|X_2\widehat{\beta_2}\|^2$$

Under what conditions will equality hold?

[$8 + 4 = 12$]