1. Consider the sequence of real numbers

$$x_n = \frac{(\alpha+1)^n + \alpha^n + (\alpha-1)^n}{(2\alpha)^n}, \; n = 1, 2, \ldots .$$

Find the real values of α for which $\lim_{n\to\infty} x_n$ exists. [10]

2. For $x \geq 0$, define $f(x) = x - \sqrt{2}\sin(x)$, with x in radians. Find the range of the function f. [10]

3. Let X_1, X_2 and X_3 be independent and identically distributed (iid) random variables with probability density function (pdf) symmetric about 0. Assuming that moments of all orders exist, find the value of

$$E\Big(\frac{X_1^3 - X_2^3 + X_3^3}{\sqrt{X_1^2 + X_2^2 + X_3^2}}\Big).$$

[10]

4. Let A be $n \times n$ non-zero matrix such that $A^2 = 0$.
(a) For $n = 2$, give an example of such a matrix A.
(b) Show that $\text{Rank}(A) \leq [\frac{n}{2}]$, where $[x]$ denotes the highest integer less than or equal to x.
(c) Give an example of such a matrix A for which the equality is attained in (b), i.e. $\text{Rank}(A) = [\frac{n}{2}]$. $[3 + 3 + 4 = 10]$

5. Let A be a real symmetric 4×4 matrix. Consider the matrix $I+A+A^2$, where I is the 4×4 identity matrix. Show that the trace (sum of the diagonal elements) of the matrix $I + A + A^2$ cannot be smaller than 3. [10]

6. If the distribution of the radii of some spheres is symmetric, what can be said about the skewness of the distribution of their volumes? You may use the Bowley's measure of skewness for this purpose, which is defined as, skewness=$\frac{Q_3-2Q_2+Q_1}{Q_3-Q_1}$, where Q_i is the i-th quartile (for $i = 1, 2, 3$). [10]

7. From a common starting point on a circular path around a park, five persons started moving in the clockwise direction with speeds $X_1, X_2, \ldots, X_5$ meters per second, respectively. Suppose that the total length of the circular path is 200 meters. Assume that $X_1, X_2, \ldots, X_5$ are independent and identically distributed as Uniform (0,2) variables, and each person takes several laps around the park. If T denotes the time for the first instance when two persons meet after starting, find the expected value of T. [10]

8. In a two-player game each player starts with an initial score zero. Player I rolls a die first and Player II randomly predicts the outcome. If the actual outcome i is different from the predicted outcome j, the score of Player I is increased by i, and Player I rolls the die again. This process is continued until Player II predicts the correct outcome. If the prediction is correct, nothing is added to the score of Player I, and the current score is considered as the final score S. Find the expectation and variance of S. [10]

9. Consider a population of N units with x denoting the size variable and y the response variable. We have observations on only those x and y values for the units in an SRSWOR (Simple Random Sampling Without Replacement) sample of size n.

   We are interested in the parameter $R = \frac{Y}{X}$, where Y and X denote the total of all the y and x values in the population. Also, the y and x values in the population are related by a perfect straight line with both being positive.

   The proposed estimator of R is

$$\bar{r} = \frac{1}{n} \sum_{i=1}^{n} \frac{y_i}{x_i}.$$

   Show that the bias of this estimator is positive. [10]

10. Consider the Markov Chain with state space $\{1, 2, 3, 4, 5, 6\}$, and transition probability matrix

$$A = \begin{bmatrix} 1/3 & 2/3 & 0 & 0 & 0 & 0 \\ 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1/4 & 1/2 & 1/4 \\ 0 & 0 & 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 0 & 1/4 & 1/4 & 1/2 \end{bmatrix}.$$

Find the set of all stationary distributions of this chain. [10]

11. Let X_i, $i = 1, 2, \ldots, 5$, be independent and identically distributed with distribution function F, and Y_j, $j = 1, 2, \ldots, 7$, be independent and identically distributed with distribution function G, where $G(x) = F(x - \Delta)$, for all $x \in \mathbb{R}$ (the set of real numbers), and for some $\Delta \in \mathbb{R}$. Assume that F is a continuous distribution function, and X_i and Y_j are independent for all i and j. Let

$$\hat{\Delta} = \text{median}\{Y_j - X_i, \ \ 1 \leq i \leq 5, \ \ 1 \leq j \leq 7\}.$$

Show that the distribution of $\hat{\Delta} - \Delta$ is the same for all $\Delta \in \mathbb{R}$. [10]

12. If the mean and the median of 10 observations $x_1, x_2, \ldots, x_{10}$ are 2 and 3, respectively, show that a minimizer of

$$\Psi(\theta) = 2 \sum_{i=1}^{10} (x_i - \theta)^2 + 3 \sum_{i=1}^{10} |x_i - \theta|$$

cannot lie outside the interval [2,3]. Is this minimizer unique? Justify your answer. [6 + 4 = 10]
