1. Consider two urns each containing a_i balls with number i written on it, for $i = 1, 2, 3$, so that the total number of balls in each urn is $(a_1 + a_2 + a_3)$ to start with. Now, a ball is drawn at random from urn 1 and let X denote the number written on it. Consequently, b balls of kind a_X are added to urn 2. Now a ball is drawn at random from urn 2 and let Y denote the number written on it.

a) Find the marginal distribution of Y.

b) Find the correlation between X and Y.

[6+9]=15

2. Let p_1 and p_2 be the success probabilities of two treatments A and B. Suppose (p_1, p_2) has a two-point prior given by

$$(p_1, p_2) = \begin{cases} (a, 1-a) & \text{with probability } \frac{1}{2}, \\ (1-a, a) & \text{with probability } \frac{1}{2}, \end{cases}$$

for some $a \in (0, 1),\ a \neq \frac{1}{2}$. Now, let s_1 and s_2 successes occur from n_1 and n_2 samples from treatment A and B, respectively. Find the posterior probability of the event $\{p_1 > p_2\}$.

[15]

3. (a) Let $X_1, X_2, \ldots$ be independent and identically distributed random variables having distribution function

$$F(x) = \frac{1}{2} + \frac{1}{\pi} \tan^{-1}\left(\frac{x}{a}\right), \qquad a > 0.$$

Is the strong law of large numbers (SLLN) applicable for $\{X_n\}$? Justify!

(b) Suppose that $\{X_n\}$, $\{Y_n\}$ are two sequences of random variables satisfying

$$\sum_{n=1}^{\infty} P(X_n \neq Y_n) < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} |Y_n| < \infty, \quad \text{almost surely.}$$

Prove that $X_n \to 0$ almost surely.

[6+9]=15

4. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed as $\mathcal{N}(\theta, 1)$ and the parameter of interest is $p = P(X_1 \leq a)$ for fixed known a.

a) Show that the UMVUE of p is $\delta_n = \Phi\left(\sqrt{\frac{n}{n-1}}(a - \overline{X})\right)$.

b) Using the delta method, or otherwise, find the asymptotic distribution of the UMVUE δ_n.

[8+7]=15

5. Let $\underline{Y}$ and S be, respectively, the mean vector and the covariance matrix based on a random sample of size n from $\mathcal{N}_p(\underline{\mu}, \Sigma)$ distribution with unknown $\underline{\mu}$ and Σ, where $p < n$. We want to test the hypothesis

$$H_0 : \underline{\mu} = 0, \quad \text{against} \quad H_1 : \underline{\mu} \neq 0.$$

For this purpose, we consider the following three test statistics.

$$\begin{aligned} T_1 &= \text{usual Hotelling's } T^2 \text{ test statistic,} \\ T_2 &= \sup_{\underline{l} \neq 0} \frac{(\underline{l}^T \underline{Y})^2}{(\underline{l}^T S \underline{l})}, \\ T_3 &= \frac{|S + \underline{Y}\,\underline{Y}^T|}{|S|}. \end{aligned}$$

Prove that the test based on T_1, T_2 and T_3 are all equivalent for testing the above hypothesis.

[15]

6. Suppose that $X_1, X_2, \ldots$ are independent and identically distributed random variables according to an unknown continuous distribution function F having unique median. We want to test the hypothesis

$$H_0 : F(0) = \frac{1}{2}, \quad \text{against} \quad H_1 : F(0) \neq \frac{1}{2}.$$

a) Suggest an appropriate test for the above hypothesis using the stopping variable
$$N = \min\left\{ n : \sum_{i=1}^{n} I(X_i > 0) = r \right\},$$
where $r(> 1)$ is a positive integer.

b) Find the mean and variance of N under H_1.

c) Find the asymptotic distribution of N under H_1 and hence comment on its asymptotic power.

[5+4+6]=15

7. Consider a randomized block design with b treatments, arranged in b blocks, each of size $b-1$, such that every treatment except the i-th treatment occurs exactly once in the i-th block, $i = 1, \ldots, b$. Let τ_i denote the effect of i-th treatment for each i.

a) Show that every pairwise contrast of the treatment effects is estimable.

b) Find the Best Linear Unbiased Estimator (BLUE) of $\tau_i - \tau_j$, for $i \neq j, \quad i, j \in \{1, \ldots, b\}$. Also find its variance.

[6+9]=15

8. Let F be a cumulative distribution function (CDF) on the real line with density f. Consider a two sample problem where $X_1, \ldots, X_n$ are independent and identically distributed with CDF F and $Y_1, \ldots, Y_m$ are independent and identically distributed with CDF F^γ for some $\gamma > 0$.

a) Show that F^γ is indeed a CDF.

b) Assuming F to be known, derive the most powerful test for $H_0 : \gamma = 1$ against $H_1 : \gamma = 1/2$.

c) If F is unknown, propose a suitable nonparametric test for the same hypothesis testing problem as in (b) and derive its null distribution.

[3+5+7]=15