1. For $n \geq 1$, X_n are independent random variables with the probability distribution:

$$X_n = \begin{cases} 0 & \text{with probability } \left(1 - \frac{1}{n}\right) \\ \frac{1}{n} & \text{with probability } \frac{1}{n}\left(1 - \frac{1}{n}\right) \\ n & \text{with probability } \frac{1}{n^2}. \end{cases}$$

   (a) Show that $\{X_n = n\}$ occurs only finitely many times with probability 1. Also show that $\{X_n = \frac{1}{n}\}$ occurs infinitely often with probability 1.

   (b) Determine if the sequence $\{X_n\}$ converges almost surely or not.

[8 + 7 = 15]

2. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(\theta, 1)$ where θ is an unknown integer.

   (a) Find the maximum likelihood estimator $\widehat{\theta}_n$ of θ.

   (b) Prove that it is not possible to find a sequence of real constants a_n such that $a_n(\widehat{\theta}_n - \theta)$ converges to a non-degenerate limiting distribution.

[7 + 8 = 15]

3. Suppose, the observable random vector $(Y_1, Y_2, \ldots, Y_5)'$ is normally distributed with mean $\Big((\alpha+\beta), (\alpha+\beta), (\alpha-\gamma), (\alpha-\gamma), (\delta-\alpha)\Big)'$ and covariance matrix $\sigma^2\mathrm{I}$ where I is the identity matrix of appropriate order, and α, β, γ, δ, σ^2 are fixed unknown parameters.

   (a) Find the class of linear unbiased estimators of $(\delta - \alpha)$.

   (b) For observed $(y_1, \ldots, y_5)' = (10, 12, 6, 10, 22)'$, find a $100(1-\theta)\%$ confidence region for $(\alpha + \beta, \alpha - \gamma)'$ in terms of a quantile of a known distribution, where $0 < \theta < 1$ is a given number.

[6 + 9 = 15]

4. Suppose, $(X_1, X_2, \ldots, X_k)$ is distributed as Multinomial $(n, p_1, p_2, \ldots, p_k)$ where $p_1, \ldots, p_k$ are positive numbers with $\sum_{i=1}^{k} p_i = 1$. Define the vector $\mathbf{p}'$ as $(p_1, p_2, \ldots, p_k)'$.

   (a) Find the asymptotic variance of the maximum likelihood estimator of $\mathbf{p}'\mathbf{p}$.

   (b) Suppose S_1 and S_2 are disjoint subsets of $\{1, 2, \ldots, k\}$ such that $d_i = |S_i| \geq 2,\ i = 1, 2$. Suggest a suitable asymptotic test of level α for

$$\begin{aligned} H_0 : &\quad p_i = p_j\ \forall\, i, j \in S_1,\ \ p_l = p_m\ \forall\, l, m \in S_2, \\ vs.\ H_1 : &\quad \text{not } H_0. \end{aligned}$$

[8 + 7 = 15]

5. Let $X_1, X_2, \ldots, X_n$ be an independent and identically distributed sample from a distribution having the cumulative distribution function G, where

$$G(x) = (1 - p)F(x) + pH(x).$$

   F and H are distinct distributions which are known, and $p \in (0, 1)$ is unknown. Based on the sample, suggest an estimator of p which is unbiased and consistent. Justify your answer.

[15]

6. Suppose that a teacher does not know the exact number of students who have opted for an online course that she is teaching. A total of 24 students were present in the first class of the course. The teacher recorded the names of those students. In her second class, she found that a total of 30 students were present, of which 16 were also present in her first class. Assuming that each student is equally likely to attend any class, independent of each other, find the most likely number of students who have opted for the course. [15]

7. We would like to test seven treatments A, B, C, D, E, F, G in blocks of three plots according to one of the following plans, the primary objective being to estimate and test the pairwise differences of treatment effects.

Plan	Block 1	Block 2	Block 3	Block 4	Block 5	Block 6	Block 7
I	ABD	CFG	AEF	BEF	ACG	DEG	BCG
II	ABC	BFD	CDG	ADE	CEF	AFG	BEG
III	ABC	ACD	ADE	AEF	AFG	ABG	ABD

(a) Which plan do you think will serve the purpose best? Give reasons.

(b) For the best design, obtain the best linear unbiased estimators of the pairwise differences of the treatment effects, along with their variances.

$[5 + 10 = 15]$

8. Suppose, we have two independent samples $\mathbf{X}$ and $\mathbf{Y}$:

$\mathbf{X}$ consists of $X_1, X_2, \ldots, X_m$ independent and identically distributed as $F(x)$;

$\mathbf{Y}$ consists of $Y_1, Y_2, \ldots, Y_n$ independent and identically distributed as $F(x - \theta)$,

where F is a continuous distribution function. We want to test $H_0 : \theta = 0$ vs. $H_1 : \theta > 0$. Suppose that we reject H_0 for large values of a rank statistic of the form $S = S(\mathbf{X}, \mathbf{Y}) = \sum_{i=1}^{n} a(R_i)$ where $a(1) \leq a(2) \leq \ldots \leq a(m + n)$ are known values, not all equal, and R_i is the rank of Y_i among all $(m + n)$ observations.

(a) Show that the test based on S has a monotone power function in θ.

(b) Suppose that the above test is of size α. Show that this test is also an unbiased test of level α for testing $H_0^* : \theta \leq 0$ vs. $H_1 : \theta > 0$.

$[10 + 5 = 15]$