1. Let $X_1, X_2, \ldots, X_m$ and $Y_1, Y_2, \ldots, Y_n$ be independent random samples, respectively, from $N(\mu, \sigma_0^2)$ and $N(2\mu, \sigma_0^2)$, where $\sigma_0 (> 0)$ is known and $-\infty < \mu < \infty$ is the unknown parameter.
(a) Find the Most Powerful level α test for testing $H_0 : \mu = 0$ vs. $H_1 : \mu = 1$.
(b) Is the test in (a) also the Uniformly Most Powerful level α test for testing
$$H_0 : \mu \leq 0 \text{ versus } H_1 : \mu > 0?$$
[10 + 5 = 15]

2. Suppose $X \sim \text{Poisson}(\lambda)$, $\lambda > 0$. Let $\theta = \lambda^3 \exp(-2\lambda)$.
(a) Show that $S = (-1)^{X-1} X(X-1)(X-2)$ is an unbiased estimate of θ.
(b) Consider a random sample $X_1, X_2, \ldots, X_n$ from Poisson(λ). Derive the Uniformly Minimum Variance Unbiased Estimator of θ.
[7 + 8 = 15]

3. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed $Uniform(0,1)$ random variables. If $Z_n = \sum_{i=1}^{n} X_i^2$, find $\lim_{n\to\infty} P\left(Z_n \leq \frac{n}{3} + \left(\frac{3}{n}\right)^{\frac{1}{4}}\right)$. [15]

4. Suppose $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_m$ is a random sample from a Trinomial distribution with $n = 1$, and $\boldsymbol{\pi} = [\pi_1, \pi_2, \pi_3]^T$, where $0 < \pi_1, \pi_2, \pi_3 < 1$, and $\pi_1 + \pi_2 + \pi_3 = 1$. Find an approximate 95% confidence interval for $\pi_1 + \frac{1}{2}\pi_2$, for large m. [15]

5. Let $Y_1, Y_2, \ldots$ be categorical random variables each having categories $C_1, C_2, \ldots, C_k$. For $t \geq 2$, the conditional probabilities are given as follows:
$P(Y_t = C_i | Y_{t-1} = C_{j_1}, Y_{t-2} = C_{j_2}, \ldots.) = P(Y_t = C_i | Y_{t-1} = C_{j_1}) = (1-\alpha)p_i + \alpha I(i = j_1)$, and $P(Y_1 = C_i) = p_i$, where $i, j_1, j_2, \ldots = 1, 2, \ldots, k$, $0 \leq \alpha \leq 1$, and $I(.)$ is an indicator.
(a) Find the marginal distribution of Y_t, for $t \geq 2$.
(b) Find the conditional distribution of Y_{t+h} given Y_t, for $h > 1$.

(c) If $C_i = i$, for all i, find the correlation coefficient between Y_t and Y_{t+h}, for $h \geq 1$. $[3 + 5 + 7 = 15]$

6. Consider a block design in v blocks of size $v + 1$ with v treatments such that the i-th treatment occurs twice in the i-th block and all other $v - 1$ treatments occur once, for $i = 1, 2, \ldots, v$. Assuming a fixed effects additive model, answer the following questions.
(a) Find a set of maximum number of estimable independent pairwise difference of treatment effects. Hence or otherwise conclude if the design is connected.
(b) Is the design orthogonal? Give reasons.
(c) Obtain the C-matrix of the block design and find the best linear unbiased estimator (BLUE) of all estimable pairwise difference of treatment effects in the above set you mentioned, along with variance of the BLUEs. $[(2 + 1) + (1 + 2) + (3 + 4 + 2) = 15]$

7. Suppose that (X, Y) follows a uniform distribution on the unit circle with the centre at the origin. Then prove or disprove the following statements.
(a) $|X|$ and $|Y|$ are independent.
(b) Sign (X) and Sign (Y) are independent, where $\text{Sign}(t) = 1, 0, -1$, respectively, for $t > 0, t = 0$, and $t < 0$.
(c) $|X|$ and Sign (X) are independent. $[5 + 5 + 5 = 15]$

8. Suppose that $\begin{bmatrix} X_1 \\ Y_1 \end{bmatrix}, \begin{bmatrix} X_2 \\ Y_2 \end{bmatrix}, \ldots, \begin{bmatrix} X_n \\ Y_n \end{bmatrix}$ are independent and identically distributed as uniform random vectors on the square with vertices $(\theta, 0), (0, \theta), (-\theta, 0), (0, -\theta)$.
(a) Find the Maximum Likelihood Estimator of θ.
(b) Show that the estimator in (a) converges to θ in probability as $n \to \infty$. $[8 + 7 = 15]$