1. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x) = \frac{2-\sqrt{x+4}}{\sin 2x}$ for all $x \neq 0$. Then the value of $f(0)$ is

(A) $-\frac{1}{8}$ (B) $\frac{1}{8}$ (C) 0 (D) $-\frac{1}{4}$

2. A person throws a pair of fair dice. If the sum of the numbers on the dice is a perfect square, then the probability that the number 3 appeared on at least one of the dice is

(A) 1/9 (B) 4/7 (C) 1/18 (D) 7/36

3. Consider the system of linear equations: $x+y+z = 5$, $2x+2y+3z = 4$. Then

(A) the system is inconsistent
(B) the system has a unique solution
(C) the system has infinitely many solutions
(D) none of the above is true

4. If $g'(x) = f(x)$ then $\int x^3 f(x^2)dx$ is given by

(A) $x^2 g(x^2) - \int xg(x^2)dx + C$ (B) $\frac{1}{2}x^2 g(x^2) - \int xg(x^2)dx + C$
(C) $2x^2 g(x^2) - \int xg(x^2)dx + C$ (D) $x^2 g(x^2) - \frac{1}{2}\int xg(x^2)dx + C$

5. If $({}^nC_0 +^n C_1)({}^nC_1 +^n C_2)\cdots({}^nC_{n-1} +^n C_n) = k\ {}^nC_0\ {}^nC_1\ \cdots\ {}^nC_{n-1}$, then k is equal to

(A) $\frac{(n+1)^n}{n!}$ (B) $\frac{n^n}{n!}$ (C) $\frac{(n+1)^n}{nn!}$ (D) $\frac{(n+1)^{n+1}}{n!}$

6. Let $\{f_n\}$ be a sequence of functions defined as follows:

$$f_n(x) = x^n \cos(2\pi nx), \quad x \in [-1, 1].$$

Then $\lim_{n\to\infty} f_n(x)$ exists if and only if x belongs to the interval

(A) $(-1, 1)$ (B) $[-1, 1)$ (C) $[0, 1]$ (D) $(-1, 1]$

7. Let S be a set of n elements. The number of ways in which n distinct non-empty subsets $X_1, ..., X_n$ of S can be chosen such that $X_1 \subseteq X_2 \cdots \subseteq X_n$, is

(A) $\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}$ (B) 1 (C) $n!$ (D) 2^n

8. Let A be a 4×4 matrix such that both A and $\mathrm{Adj}(A)$ are non-null. If $\det A = 0$, then the rank of A is

(A) 1 (B) 2 (C) 3 (D) 4

9. The set of all a satisfying the inequality

$$\frac{1}{\sqrt{a}} \int_1^a \left(\frac{3}{2}\sqrt{x} + 1 - \frac{1}{\sqrt{x}}\right) dx < 4$$

is equal to the interval

(A) $(-5, -2)$ (B) $(1, 4)$ (C) $(0, 2)$ (D) $(0, 4)$

10. Let C_0 be the set of all continuous functions $f : [0,1] \to \mathbb{R}$ and C_1 be the set of all differentiable functions $g : [0,1] \to \mathbb{R}$ such that the derivative g' is continuous. (Here, differentiability at 0 means right differentiability and differentiability at 1 means left differentiability.) If $T : C_1 \to C_0$ is defined by $T(g) = g'$, then

(A) T is one-to-one and onto
(B) T is one-to-one but not onto
(C) T is onto but not one-to-one
(D) T is neither one-to-one nor onto.

11. Suppose a, b, c are in A.P. and a^2, b^2, c^2 are in G.P. If $a < b < c$ and $a + b + c = \frac{3}{2}$, then the value of a is

(A) $\frac{1}{2\sqrt{2}}$ (B) $-\frac{1}{2\sqrt{2}}$ (C) $\frac{1}{2} - \frac{1}{\sqrt{3}}$ (D) $\frac{1}{2} - \frac{1}{\sqrt{2}}$

12. The number of distinct even divisors of

$$\prod_{k=1}^{5} k!$$

is

(A) 24 (B) 32 (C) 64 (D) 72

13. Let D be the triangular region in the xy-plane with vertices at $(0,0),(0,1)$ and $(1,1)$. Then the value of

$$\int\int_D \frac{2}{1+x^2}\,dx\,dy$$

is

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{2}-\ln 2$ (C) $2\ln 2$ (D) $\ln 2$

14. Given a real number $\alpha \in (0,1)$, define a sequence $\{x_n\}_{n\geq 0}$ by the following recurrence relation:

$$x_{n+1} = \alpha x_n + (1-\alpha)x_{n-1}, \quad n \geq 1.$$

If $\lim_{n\to\infty} x_n = \ell$ then the value of ℓ is

(A) $\frac{\alpha x_0 + x_1}{1-\alpha}$ (B) $\frac{(1-\alpha)x_0 + x_1}{2-\alpha}$

(C) $\frac{\alpha x_0 + x_1}{2-\alpha}$ (D) $\frac{(1-\alpha)x_1 + x_0}{2-\alpha}$

15. A straight line passes through the intersection of the lines given by $3x - 4y + 1 = 0$ and $5x + y = 1$ and makes equal intercepts of the same sign on the coordinate axes. The equation of the straight line is

(A) $23x - 23y + 11 = 0$ (B) $23x - 23y - 11 = 0$

(C) $23x + 23y + 11 = 0$ (D) $23x + 23y - 11 = 0$

16. The series

$$\sum_n \frac{3 \cdot 6 \cdot 9 \cdots 3n}{7 \cdot 10 \cdot 13 \cdots (3n+4)} x^n, \quad x > 0$$

(A) converges for $0 < x \leq 1$ and diverges for $x > 1$

(B) converges for all $x > 0$

(C) converges for $0 < x < \frac{1}{2}$ and diverges for $x \geq \frac{1}{2}$

(D) converges for $\frac{1}{2} < x < 1$ and diverges for $0 < x \leq \frac{1}{2}$, $x \geq 1$.

17. Suppose A and B are two square matrices such that the largest eigen-value of $(AB - BA)$ is positive. Then the smallest eigen value of $(AB - BA)$

(A) must be positive
(B) must be negative
(C) must be 0
(D) is none of the above

18. The number of saddle points of the function $f(x, y) = 2x^4 - x^2 + 3y^2$ is

(A) 1
(B) 0
(C) 2
(D) none of the above

19. Suppose G is a cyclic group and $a, b \in G$. There does not exist any $x \in G$ such that $x^2 = a$. Also, there does not exist an $y \in G$ such that $y^2 = b$. Then,

(A) there exists an element $g \in G$ such that $g^2 = ab$.

(B) there exists an element $g \in G$ such that $g^3 = ab$.

(C) the smallest exponent $k > 1$ such that $g^k = ab$ for some $g \in G$ is 4.

(D) none of the above is true.

20. The number of real roots of the polynomial $x^3 - 2x + 7$ is

(A) 0
(B) 1
(C) 2
(D) 3

21. Suppose that a 3×3 matrix A has an eigen value -1. If the matrix $A + I$ is equal to

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

then the eigen vectors of A corresponding to the eigenvalue -1 are in the form,

(A) $\begin{bmatrix} 2t \\ 0 \\ t \end{bmatrix}, \quad t \in \mathbb{R}$

(B) $\begin{bmatrix} 2t \\ s \\ t \end{bmatrix}, \quad s, t \in \mathbb{R}$

(C) $\begin{bmatrix} t \\ 0 \\ -2t \end{bmatrix}, \quad t \in \mathbb{R}$

(D) $\begin{bmatrix} t \\ s \\ 2t \end{bmatrix}, \quad s, t \in \mathbb{R}$

22. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f(x, y) = x^2(y - 1)$. For $\vec{u} = (\frac{1}{2}, \frac{1}{2})$ and $\vec{v} = (3, 4)$, the value of the limit

$$\lim_{t \to 0} \frac{f(\vec{u} + t\vec{v}) - f(\vec{u})}{t}$$

is

(A) $\frac{3}{4}$ (B) $\frac{6}{13}$ (C) $-\frac{1}{2}$ (D) none of the above

23. Suppose ϕ is a solution of the differential equation $y'' - y' - 2y = 0$ such that $\phi(0) = 1$ and $\phi'(0) = 5$. Then

(A) $\phi(x) \to \infty$ as $|x| \to \infty$

(B) $\phi(x) \to -\infty$ as $|x| \to \infty$

(C) $\phi(x) \to -\infty$ as $x \to -\infty$

(D) $\phi(x) \to \infty$ as $x \to -\infty$

24. A fair die is rolled five times. What is the probability that the largest number rolled is 5?

(A) 5/6 (B) 1/6 (C) $1 - (1/6)^6$ (D) $(5/6)^5 - (2/3)^5$

25. Two rows of n chairs, facing each other, are laid out. The number of different ways that n couples can sit on these chairs such that each person sits directly opposite to his/her partner is

(A) $n!$ (B) $n!/2$ (C) $2^n n!$ (D) $2n!$.

26. Consider the function $f : \mathbb{C} \to \mathbb{C}$ defined on the complex plane $\mathbb{C}$ by $f(z) = e^z$. For a real number $c > 0$, let $A = \{f(z) | \text{ Re } z = c\}$ and $B = \{f(z) | \text{ Im } z = c\}$. Then

(A) both A and B are straight line segments
(B) A is a circle and B is a straight line segment
(C) A is a straight line segment and B is a circle
(D) both A and B are circles

27. Consider two real valued functions f and g given by

$$f(x) = \frac{x}{x-1} \text{ for } x > 1, \quad \text{and} \quad g(x) = 7 - x^3 \text{ for } x \in \mathbb{R}.$$

Which of the following statements about inverse functions is true?

(A) Neither f^{-1} nor g^{-1} exists
(B) f^{-1} exists, but not g^{-1}
(C) f^{-1} does not exist, but g^{-1} does
(D) Both f^{-1} and g^{-1} exist.

28. A circle is drawn with centre at $(-1, 1)$ touching $x^2+y^2-4x+6y-3 = 0$ externally. Then the circle touches

(A) both the axes (B) only the x-axis
(C) none of the two axes (D) only the y-axis

29. Let $f(x-y) = \frac{f(x)}{f(y)}$ for all $x, y \in \mathbb{R}$ and $f'(0) = p, f'(5) = q$. Then the value of $f'(-5)$ is

(A) q (B) $-q$ (C) $\frac{p}{q}$ (D) $\frac{p^2}{q}$

30. Let

$$A = \begin{bmatrix} a & 1 & 1 \\ b & a & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

The number of elements in the set

$$\left\{(a, b) \in \mathbb{Z}^2 : 0 \leq a, b \leq 2021,\ \text{rank}(A) = 2\right\}$$

is

(A) 2021 (B) 2020 (C) $2021^2 - 1$ (D) 2020×2021