1. Let C be the circle in the xy-plane which passes through the origin and touches the line $y = -2$. If the centre of C lies on the straight line $x + y = 1$, then the radius of C is

(A) $6 \pm \sqrt{12}$. (B) $5 \pm \sqrt{12}$. (C) $4 \pm \sqrt{12}$. (D) $\sqrt{12} \pm 3$.

2. The set of positive real numbers x for which the series

$$\sum_{n=1}^{\infty} \frac{1}{x^n + x^{-n}}$$

diverges

(A) is the empty set.
(B) is a set having exactly 1 element.
(C) is a finite set having at least 2 elements.
(D) is an infinite set.

3. Let X be a non-empty set and let $\mathcal{P}(X)$ be its power set. Define two operations '$+$' and '$*$' on $\mathcal{P}(X)$ as follows. For $A, B \in \mathcal{P}(X)$, let

$$A + B = A \cup B\,, \qquad A * B = A \cap B\,.$$

If $G_1 = (\mathcal{P}(X), +)$ and $G_2 = (\mathcal{P}(X), *)$, then,

(A) G_1 and G_2 are groups.
(B) G_1 is a group but G_2 is not.
(C) G_2 is a group but G_1 is not.
(D) neither G_1 nor G_2 is a group.

4. The system of equations

$$\begin{aligned} x + 2y + 3z &= c \\ 2x - y + z &= 2 \\ ax + y + 4z &= b \end{aligned}$$

has no solution if

(A) $a \neq 3$. (B) $b = c + 2$.

(C) $a = 3, b \neq c + 2$. (D) $a = 3, b = c + 2$.

5. Let

$$f(x) = \begin{cases} 4x - 1, & \text{if } x \leq 2\,, \\ x^2, & \text{if } x > 2\,. \end{cases}$$

At $x = 2$,

(A) f is differentiable.

(B) the left and right derivatives of f exist and are unequal.

(C) the left derivative of f exists and the right derivative does not.

(D) the right derivative of f exists and the left derivative does not.

6. What is the sum of the maximum and the minimum values of

$$\frac{1}{1 + (2\cos x - 4\sin x)^2}$$

over $x \in \mathbb{R}$?

(A) $\frac{19}{21}$ (B) $\frac{22}{21}$ (C) $\frac{23}{22}$ (D) $\frac{25}{23}$

7. Suppose that $f : A \to B$ is a function, where A and B are sets having 10 and 20 elements, respectively. Let us denote for all subsets U of A,

$$f(U) = \{f(a) : a \in U\},$$

and for all subsets V of B,

$$f^{-1}(V) = \{a \in A : f(a) \in V\}.$$

Which of the following is true?

(A) $f^{-1}(f(A)) = A$ and $f(f^{-1}(B)) = B$
(B) $f^{-1}(f(A)) = A$ and $f(f^{-1}(B)) \neq B$
(C) $f^{-1}(f(A)) \neq A$ and $f(f^{-1}(B)) = B$
(D) $f^{-1}(f(A)) \neq A$ and $f(f^{-1}(B)) \neq B$

8. Let

$$P(x) = \sum_{i=0}^{n} a_i x^i, \, x \in \mathbb{R},$$

where $n \geq 5$ and $a_0, \ldots, a_n \in \mathbb{R}$. If

$$\lim_{x \to 0} \frac{P(x) - e^x}{x^4} = 0,$$

then

(A) $a_4 = 0$.
(B) $a_4 = \frac{1}{24}$.
(C) $a_5 = 0$.
(D) $a_5 = \frac{1}{120}$.

9. For all $x \in \mathbb{R}$, let

$$P(x) = x^d + \sum_{i=0}^{d-1} a_i x^i \,, \qquad \text{and} \qquad Q(x) = x^d + \sum_{i=0}^{d-1} b_i x^i \,,$$

where $d \geq 1$ and $a_0, \ldots, a_{d-1}, b_0, \ldots, b_{d-1} \in \mathbb{R}$. Suppose that $n \geq d$ and $x_1, \ldots, x_n$ are distinct real numbers. Let $C(P, Q)$ be the number of elements in the set

$$\{x_i : P(x_i) \neq Q(x_i)\} \ .$$

Which of the following conditions ensures that P and Q are identical?

(A) $C(P, Q) \leq 1$
(B) $C(P, Q) \leq d - 1$
(C) $C(P, Q) \leq n - d$
(D) None of the above

10. For any positive integer n, the value of

$$\sum_{j=0}^{n} (-1)^j \binom{2n}{2j}$$

is

(A) 0 .
(B) $2^n \sin(n\pi/2)$.
(C) $2^{n/2} \cos(n\pi/2)$.
(D) $2^n \cos(n\pi/2)$.

11. The set of all solutions of the equation

$$z^3 + 8i = 0 \,,$$

where $i = \sqrt{-1}$ is

(A) $\{2i\}$.
(B) $\{2\sqrt{2}i\}$.
(C) $\{2i, (\sqrt{5}-1)i, (-\sqrt{5}-1)i\}$.
(D) $\{2i, \sqrt{3} - i, -\sqrt{3} - i\}$.

12. A lie detector determines correctly whether a person is lying or telling the truth with probability 0.8, independently of whether the person is actually lying or not. Assume that people tell the truth 90% of the times. Given that for a particular person the lie detector says that the person is telling the truth, what is the conditional probability that he is actually telling the truth?

(A) 0.9 (B) 0.8 (C) $\frac{36}{37}$ (D) $\frac{76}{77}$

13. Let $f(x, y) = ax^2 + by^2 - c$, for $(x, y) \in \mathbb{R}^2$, where $a, b, c \in \mathbb{R}$ with $bc > 0$ and $a \neq 0$. Let

$$S = \{(x, y) \in \mathbb{R}^2 : f(x, y) = 0\}.$$

Consider the map $p : \mathbb{R}^2 \to \mathbb{R}$ defined by $p(x, y) = x$. If

$$p(S) = \{p(x, y) : (x, y) \in S\},$$

then which of the following can never be true?

(A) $p(S) = \mathbb{R}$.
(B) $p(S)$ is a closed bounded interval.
(C) $p(S)$ is an open bounded interval.
(D) For all $z \in p(S)$, $-z$ also belongs to $p(S)$.

14. For $\alpha \in \mathbb{R}$ with $\alpha \neq 0$,

$$\int_{-\infty}^{\infty} x e^{\alpha x - 2|\alpha x|}\, dx$$

equals

(A) $\frac{10}{9}\alpha^{-2}$. (B) $\frac{10}{9}(\alpha|\alpha|)^{-1}$.
(C) $\frac{8}{9}\alpha^{-2}$. (D) $\frac{8}{9}(\alpha|\alpha|)^{-1}$.

15. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable non-decreasing function whose derivative f' is continuous. Fix $a < b$, and let

$$S = \{x \in (a,b) : f'(x) = 0\} .$$

If $f(a) < f(b)$, then which of the following necessarily holds?

(A) The set S is empty.
(B) The set S is finite and non-empty.
(C) The set S is countably infinite.
(D) None of the above.

16. Let S be the set of all continuous functions $f : [0,\infty) \to \mathbb{R}$ satisfying the equation

$$\int_0^x f(t)\,dt = f(x) - 2\,, \text{ for all } x > 0\,.$$

Then,

(A) S is an infinite set.
(B) S has exactly one element.
(C) S is a finite set containing more than one element.
(D) S is the empty set.

17. Let $A = (3,1)$, $B = (4,3)$ and $C = (4,4)$ be three points in the coordinate plane with origin O. Let D be an arbitrary point on the line segment OC. What is the largest possible area of $\triangle ABD$?

(A) $\frac{1}{2}$ (B) 1 (C) $\frac{5}{2}$ (D) 5

18. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function satisfying

$$f(x) + f'(x) = \sin x + \cos x \text{ for all } x \in \mathbb{R},$$

where f' is the derivative of f. Then, f is a bounded function if and only if

(A) $f(0) = 0$. (B) $f(0) \leq 0$. (C) $f(0) \geq 0$. (D) $f(0) > 0$.

19. If A is a 4×4 real matrix with at least 2 non-zero entries, then which of the following necessarily holds?

(A) At least one eigenvalue of A is non-zero.
(B) At least one eigenvalue of A is real.
(C) Rank of A is at least 2.
(D) None of the above.

20. If α and β are the two roots of the quadratic equation

$$x^2 + x + 1 = 0,$$

then the value of $\alpha^{2022} + \beta^{2022}$ is

(A) -1 . (B) 0 . (C) 1 . (D) 2 .

21. Let

$$\mathbb{Z}_{100} = \{0, 1, 2, \ldots, 99\}$$

be the additive group of integers modulo 100. How many subgroups does $\mathbb{Z}_{100}$ have which do not contain 10?

(A) 5 (B) 4 (C) 3 (D) 2

22. If $\sqrt[3]{x+5} - \sqrt[3]{x-5} = 1$, then what is the value of x^2 ?

(A) 50 (B) 51 (C) 52 (D) 53

23. How many natural numbers n are there such that $100 \leq n \leq 999$ and the sum of the digits of n is divisible by 5?

(A) 180 (B) 120 (C) 200 (D) 90

24. In how many ways can 12 identical balls be thrown into 5 distinguishable boxes such that no box gets an odd number of balls?

(A) $\binom{10}{4}$ (B) $\binom{11}{5}$ (C) 5^6 (D) 6^5

25. The range of the function

$$f(x) = \left(x^2 + \frac{1}{2}\right) e^{-x^2},\ x \in (-\infty, \infty),$$

is

(A) $\left(0, e^{-1/2}\right]$. (B) $\left[0, e^{-1/2}\right]$.
(C) $\left(\frac{1}{2}, e^{-1/2}\right]$. (D) $\left(0, \frac{3}{4}\right]$.

26. An examination centre has 50 rooms. In each room, 10 candidates have been assigned. Each candidate is absent with probability p, where $0 < p < 1$, independently of the other candidates. What is the probability that at least one room will have full attendance?

(A) $(1-p)^{500}$
(B) $(1-p^{10})^{50}$
(C) $[1-(1-p)^{10}]^{50}$
(D) $1-[1-(1-p)^{10}]^{50}$

27. Suppose Δ is the bounded region in the positive quadrant of the xy-plane enclosed by the x-axis, the straight line $y = x$ and the curve $x^2 + y^2 = \pi/2$. Then the integral

$$\iint_{\Delta} \sin(x^2 + y^2)\, dx\, dy$$

equals

(A) $\frac{\pi}{8}$.
(B) $\frac{\pi}{8}\left(1 - \cos\frac{\pi^2}{4}\right)$.
(C) $\frac{\pi}{4}$.
(D) $\frac{\sqrt{2}-1}{4}$.

28. A sequence of real numbers $\{a_n\}$ has a property P if and only if for every $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| < \varepsilon$ for all $n \geq N$. Then, $\{a_n\}$ does not have the property P if and only if

(A) there exists $\varepsilon > 0$ such that for all $N \in \mathbb{N}$ there exists $n \geq N$ satisfying $|a_n| \geq \varepsilon$.

(B) for all $\varepsilon > 0$ and $N \in \mathbb{N}$ there exists $n \geq N$ such that $|a_n| \geq \varepsilon$.

(C) there exists $\varepsilon > 0$ and $N \in \mathbb{N}$ such that $|a_n| \geq \varepsilon$ for all $n \geq N$.

(D) for all $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|a_n| \geq \varepsilon$ for some $n \geq N$.

29. How many ordered pairs (x, y) of positive integers satisfy the equation

$$x^2 - 2y^2 = 1\,,$$

where y is a prime number?

(A) 0 (B) 1 (C) 2 (D) 4

30. Let

$$S = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : \sum_{i=1}^{4} x_i^2 = 1 \right\},$$

and

$$m = \min_{(x_1,\ldots,x_4)\in S} \sum_{i=1}^{4} |x_i|\,, \qquad M = \max_{(x_1,\ldots,x_4)\in S} \sum_{i=1}^{4} |x_i|\,.$$

Then, m/M equals

(A) 0 . (B) $\frac{1}{4}$. (C) $\frac{1}{2}$. (D) $\frac{3}{4}$.