TEST CODE: PMB

SYLLABUS

Countable and uncountable sets;
equivalence relations and partitions;
convergence and divergence of sequence and series;
Cauchy sequence and completeness;
Bolzano-Weierstrass theorem;
continuity, uniform continuity, differentiability, Taylor Expansion;
partial and directional derivatives, Jacobians;
integral calculus of one variable – existence of Riemann integral,
fundamental theorem of calculus, change of variable, improper integrals;
elementary topological notions for metric spaces – open, closed and
compact sets, connectedness, continuity of functions;
sequence and series of functions;
elements of ordinary differential equations.

Vector spaces, subspaces, basis, dimension, direct sum;
matrices, systems of linear equations, determinants;
diagonalization, triangular forms;
linear transformations and their representation as matrices;
groups, subgroups, quotient groups, homomorphisms, products,
Lagrange's theorem, Sylow's theorems;
rings, ideals, maximal ideals, prime ideals, quotient rings,
integral domains, Chinese remainder theorem, polynomial rings, fields.

SAMPLE QUESTIONS

$\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote respectively the set of all real numbers, set of all complex numbers, set of all integers and set of all positive integers.

1. Let k be a field and $k[x, y]$ denote the polynomial ring in the two variables x and y with coefficients from k. Prove that for any $a, b \in k$ the ideal generated by the linear polynomials $x - a$ and $y - b$ is a maximal ideal of $k[x, y]$.

2. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation. Show that there is a line L such that $T(L) = L$.

3. Let $A \subseteq \mathbb{R}^n$ and $f : A \to \mathbb{R}^m$ be a uniformly continuous function. If $\{x_n\}_{n\geq 1} \subseteq A$ is a Cauchy sequence then show that $\lim_{n\to\infty} f(x_n)$ exists.

4. Let $N > 0$ and let $f : [0, 1] \to [0, 1]$ be denoted by $f(x) = 1$ if $x = 1/i$ for some integer $i \leq N$ and $f(x) = 0$ for all other values of x. Show that f is Riemann integrable.

5. Let $F : \mathbb{R}^n \to \mathbb{R}$ be defined by

$$F(x_1, x_2, \ldots, x_n) = \max\{|x_1|, |x_2|, \ldots, |x_n|\}.$$

   Show that F is a uniformly continuous function.

6. Show that every isometry of a compact metric space into itself is onto.

7. Let $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$ and $f : [0,1] \to \mathbb{C}$ be continuous with $f(0) = 0, f(1) = 2$. Show that there exists at least one t_0 in $[0,1]$ such that $f(t_0)$ is in $\mathbb{T}$.

8. Let f be a continuous function on $[0,1]$. Evaluate

$$\lim_{n\to\infty} \int_0^1 x^n f(x)dx.$$

9. Find the most general curve whose normal at each point passes though $(0,0)$. Find the particular curve through $(2,3)$.

10. Suppose f is a continuous function on $\mathbb{R}$ which is periodic with period 1, that is, $f(x+1) = f(x)$ for all x. Show that

    (i) the function f is bounded above and below,

    (ii) it achieves both its maximum and minimum and

    (iii) it is uniformly continuous.

11. Let $A = (a_{ij})$ be an $n \times n$ matrix such that $a_{ij} = 0$ whenever $i \geq j$. Prove that A^n is the zero matrix.

12. Determine the integers n for which $\mathbb{Z}_n$, the set of integers modulo n, contains elements x, y so that $x + y = 2,\ 2x - 3y = 3$.

13. Let a_1, b_1 be arbitrary positive real numbers. Define

$$a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \sqrt{a_n b_n}$$

    for all $n \geq 1$. Show that a_n and b_n converge to a common limit.

14. Show that the only field automorphism of $\mathbb{Q}$ is the identity. Using this prove that the only field automorphism of $\mathbb{R}$ is the identity.

15. Consider a circle which is tangent to the y-axis at 0. Show that the slope at any point (x, y) satisfies $\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$.

16. Consider an $n \times n$ matrix $A = (a_{ij})$ with $a_{12} = 1, a_{ij} = 0\ \forall\ (i,j) \neq (1,2)$. Prove that there is no invertible matrix P such that PAP^{-1} is a diagonal matrix.

17. Let G be a nonabelian group of order 39. How many subgroups of order 3 does it have?

18. Let $n \in \mathbb{N}$, let p be a prime number and let $\mathbb{Z}_{p^n}$ denote the ring of integers modulo p^n under addition and multiplication modulo p^n. Let $f(x)$ and $g(x)$ be polynomials with coefficients from the ring $\mathbb{Z}_{p^n}$ such that $f(x) \cdot g(x) = 0$. Prove that $a_i b_j = 0 \ \forall \ i, j$ where a_i and b_j are the coefficients of f and g respectively.

19. For any irrational number α such that $\alpha^2 \in \mathbb{N}$, we define $\mathbb{Q}(\alpha) := \{a + b\alpha : a, b \in \mathbb{Q}\}$. Show that $\mathbb{Q}(\alpha)$ is a field.

20. Show that the fields $\mathbb{Q}(\sqrt{2})$ and $\mathbb{Q}(\sqrt{3})$ are isomorphic as $\mathbb{Q}$-vector spaces but not as fields.

21. Suppose $a_n \geq 0$ and $\sum a_n$ is convergent. Show that $\sum 1/(n^2 a_n)$ is divergent.

22. Show that $\int_0^\infty \left| \frac{\sin x}{x} \right| dx = \infty$.

23. Suppose we have a sequence of continuous functions $f_n : [0, 1] \to \mathbb{R}$, $n \geq 1$ and another continuous function $f : [0, 1] \to \mathbb{R}$. Show that $\{f_n\}$ converges uniformly to f if and only if $f_n(x_n) \to f(x)$ whenever $x_n \to x$.

24. Let G be a group which has only finitely many subgroups. Prove that G must be a finite group.

25. If (a_n) is a sequence in (0,1), then show that $\frac{1}{n} \sum_{k=1}^{n} a_k \to 0$ if and only if $\frac{1}{n} \sum_{k=1}^{n} a_k^2 \to 0$.

26. Prove that the largest possible number of $1'$s in an $n \times n$ invertible matrix with all entries 0 or 1 is $n^2 - n + 1$.

27. Let A be a commutative ring with unity. Prove that the set

$$\{a \in A : ab = 0 \text{ for some nonzero } b \in A\}$$

contains a prime ideal of A.

MODEL QUESTION PAPER

- Please answer FOUR questions from EACH group.
- Each question carries 10 marks. Total marks : 80.
- $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote respectively the set of all real numbers, set of all complex numbers, set of all integers and set of all positive integers.

Group A

1. Let f be a twice differentiable function on $(0, 1)$. It is given that for all $x \in (0, 1)$, $|f''(x)| \leq M$ where M is a non-negative real number. Prove that f is uniformly continuous on $(0, 1)$.

2. Let f be a real-valued continuous function on $[0, 1]$ which is twice continuously differentiable on $(0, 1)$. Suppose that $f(0) = f(1) = 0$ and f satisfies the following equation:

$$x^2 f''(x) + x^4 f'(x) - f(x) = 0.$$

   (a) If f attains its maximum M at some point x_0 in the open interval $(0, 1)$, then prove that $M = 0$.

   (b) Prove that f is identically zero on $[0, 1]$.

3. Consider the set S consisting of all Cauchy sequences $(a_n)_{n \in \mathbb{N}}$ with $a_n \in \mathbb{N}$ for all n. Is the set S countable? Justify your answer.

4. Let A be a compact subset of $\mathbb{R} \setminus \{0\}$ and B be a closed subset of $\mathbb{R}^n$. Prove that the set $\{a \cdot b \mid a \in A, b \in B\}$ is closed in $\mathbb{R}^n$.

5. Does there exist a continuous function $f : [0, 1] \to [0, \infty)$ such that $\displaystyle\int_0^1 x^n f(x)\, dx = 1$ for all $n \geq 1$? Justify your answer.

6. Prove that there exists a constant $c > 0$ such that for all $x \in [1, \infty)$,

$$\sum_{n \geq x} \frac{1}{n^2} \leq \frac{c}{x}.$$

Group B

1. Let $(\mathbb{Q}, +)$ be the group of rational numbers under addition. If G_1, G_2 are nonzero subgroups of $(\mathbb{Q}, +)$, then prove that $G_1 \cap G_2 \neq \{0\}$.

2. With proper justifications, examine whether there exists any surjective group homomorphism

(a) from the group $(\mathbb{Q}(\sqrt{2}), +)$ to the group $(\mathbb{Q}, +)$,

(b) from the group $(\mathbb{R}, +)$ to the group $(\mathbb{Z}, +)$.

3. Consider the ring

$$R = \left\{ \frac{2^k m}{n} \;\middle|\; m, n \text{ odd integers; } k \text{ is a non-negative integer} \right\}.$$

(a) Describe all the units (invertible elements) of R.

(b) Demonstrate one nonzero proper ideal I of R.

(c) Examine whether the ideal I that you have chosen, is a prime ideal of R (that is, whether $a \cdot b \in I$ implies $a \in I$ or $b \in I$).

4. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation such that $T^2 = 0$. If r denotes the rank of T (that is, $r = \dim(\text{Image}(T))$), then show that $r \leq \frac{n}{2}$.

5. Let A be a 2×2 matrix with real entries such that $\text{Tr}(A) = 0$ and $\det(A) = -1$.

(a) Prove that there is a basis of $\mathbb{R}^2$ consisting of eigenvectors of A.

(b) Suppose that T is a 2×2 real matrix with respect to the above basis such that $TA = AT$. Prove that T is a diagonal matrix with respect to that basis.

6. Let $i = \sqrt{-1}$ and $\alpha = i + \sqrt{2}$. Construct a polynomial $f(x)$ with integer coefficients such that $f(\alpha) = 0$.