INSTRUCTIONS FOR CANDIDATES

- Please answer FOUR questions from EACH group.
- Each question carries 10 marks. Total marks : 80.
- $\mathbb{R}, \mathbb{C}, \mathbb{Z}$ and $\mathbb{N}$ denote respectively the set of real numbers, set of complex numbers, set of all integers and set of all positive integers.

Group A

1. Let $a_n \in \mathbb{R}$, such that $\sum\limits_{n=1}^{\infty} |a_n| = \infty$ and $\sum\limits_{n=1}^{m} a_n \to a \in \mathbb{R}$ as $m \to \infty$. Let $a_n^+ = \max\{a_n, 0\}$. Show that $\sum_{n=1}^{\infty} a_n^+ = \infty$.

2. Let $E = \{(x, y, z) \in \mathbb{R}^3 \mid x, y, z > 0, xy + yz + zx = 1\}$. Prove that there exists $(a, b, c) \in E$ such that $abc \geq xyz$, for all $(x, y, z) \in E$.

3. Let $f\colon \mathbb{R} \to \mathbb{R}$ be an increasing function. Suppose there are sequences (x_n) and (y_n) such that $x_n < 0 < y_n$ for all $n \geq 1$ and $f(y_n) - f(x_n) \to 0$ as $n \to \infty$. Prove that f is continuous at 0.

4. Do there exist continuous functions P and Q on $[0, 1]$ such that $y(t) = \sin(t^2)$ is a solution to $y'' + Py' + Qy = 0$ on $[\frac{1}{n}, 1]$ for all $n \geq 1$? Justify your answer.

5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by
$$f(x) = \int_{e^{x^3+x}}^{1+e^{x^3+x}} e^{r^2}\, dr$$
for all $x \in \mathbb{R}$. Prove that f is monotone.

6. Let $w = \{w(i, j)\}_{1 \leq i,j \leq m}$ be an $m \times m$ symmetric matrix with non-negative real entries such that $w(i, j) = 0$ if and only if $i = j$. Show that $d(i, j) = \min\{\sum\limits_{j=0}^{k-1} w(i_j, i_{j+1}) \mid k \geq 1, i_0 = i, i_k = j,\ i_j \in \{1, ..., m\}\}$ is a metric on $\{1, ..., m\}$.

Group B

7. Factory A produces 1 bad watch in 100 and factory B produces 1 bad watch in 200. You are given two watches from one of the factories and you don't know which one.

   (a) What is the probability that the second watch works?

   (b) Given that the first watch works, what is the probability that the second watch works?

8. Let R be a commutative ring containing a field k as a sub-ring. Assume that R is a finite dimensional k-vector space. Show that every prime ideal of R is maximal.

9. Let p, q be prime numbers and $n \in \mathbb{N}$ such that $p \nmid n - 1$. If $p \mid n^q - 1$ then show that $q \mid p - 1$.

10. Determine all finite groups which have exactly 3 conjugacy classes.

11. Let F be a field, $a \in F$, p a prime integer. Suppose the polynomial $x^p - a$ is reducible in $F[x]$. Prove that this polynomial has a root in F.

12. Let V be a finite-dimensional vector space over a field F and let $T : V \to V$ be a linear transformation. Let $W \subseteq V$ be a subspace such that $T(W) \subseteq W$. Suppose T is diagonalizable. Is T restricted to W also diagonalizable?