Notation. In the following, $\mathbb{R}$ denotes the set of real numbers.

(1) (a) Let $\{f_n\}$ be a sequence of continuous real-valued functions on $[0,1]$ converging uniformly on $[0,1]$ to a function f. Suppose for all $n \geq 1$ there exists $x_n \in [0,1]$ such that $f_n(x_n) = 0$. Show that there exists $x \in [0,1]$ such that $f(x) = 0$.

(b) Give an example of a sequence $\{f_n\}$ of continuous real-valued functions on $[0,\infty)$ converging uniformly on $[0,\infty)$ to a function f, such that for each $n \geq 1$ there exists $x_n \in [0,\infty)$ satisfying $f_n(x_n) = 0$, but f satisfies $f(x) \neq 0$ for all $x \in [0,\infty)$.

(2) Let $f : [0,1] \to \mathbb{R}$ be a continuous function. Show that

$$\lim_{n\to\infty} \prod_{k=1}^{n} \left(1 + \frac{1}{n} f\left(\frac{k}{n}\right)\right) = e^{\int_0^1 f(x)dx}.$$

(3) (a) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Show that

$$\lim_{h\to 0} \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} = f''(x)$$

for all $x \in \mathbb{R}$.

(b) Show that if f further satisfies

$$\frac{1}{2y}\int_{x-y}^{x+y} f(t)dt = f(x)$$

for all $x \in \mathbb{R}, y > 0$, then there exist $a, b \in \mathbb{R}$ such that $f(x) = ax + b$ for all $x \in \mathbb{R}$.

(4) Let $f : \mathbb{R} \to \mathbb{R}$ be a twice continuously differentiable function. Show that if f is bounded and $f''(x) \geq 0$ for all $x \in \mathbb{R}$ then f must be constant.

(5) Let J be a 2×2 real matrix such that $J^2 = -I$, where I is the identity matrix.

(a) Show that if $v \in \mathbb{R}^2$ and $v \neq 0$, then the vectors $v, Jv \in \mathbb{R}^2$ are linearly independent.

(b) Show that there exists an invertible 2×2 real matrix U such that

$$UJU^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

(6) Suppose V is a 3-dimensional real vector space and $T : V \to V$ is a linear map such that $T^3 = 0$ and $T^2 \neq 0$.

(a) Show that there exists a vector $v \in V$ such that the set $\{v, T(v), T^2(v)\}$ is a basis of V.

(b) Suppose $S : V \to V$ is another linear map such that $S^3 = 0$ and $S^2 \neq 0$. Show that there exists an invertible linear map $U : V \to V$ such that $S = UTU^{-1}$.

(7) Let K be a field, and let R be the ring $K[x]$. Let $I \subset R$ be the ideal generated by $(x-1)(x-2)$. Find all maximal ideals of the ring R/I.

(8) Let G be a finite group, and let H be a normal subgroup of G. Let P be a Sylow p-subgroup of H.

(a) Show that for all $g \in G$, there exists $h \in H$ such that $gPg^{-1} = hPh^{-1}$.

(b) Let $N = \{g \in G | gPg^{-1} = P\}$. Let HN be the set $HN = \{hn | h \in H, n \in N\}$. Show that $G = HN$.