Group A

1. Answer the following questions.

(a) Find all maxima and minima of the function $f(x, y) = xy$, subject to the constraints $x + 4y = 120$ and $x, y \geq 0$.

(b) Find the points on the circle $x^2 + y^2 = 50$ which are closest to and farthest from the point $(1,1)$.

(c) For what values of α are the vectors $(0,1,\alpha)$, $(\alpha, 1,0)$ and $(1, \alpha, 1)$ in $\mathcal{R}^3$ linearly independent?

[10+15+5]

2. Let $f: \mathcal{R} \rightarrow \mathcal{R}$ be a continuous function.

(a) Let Q denote the set of rational numbers. Prove that, if $f(Q) \subseteq \{1,2,3, \ldots\}$, then f is a constant function.

(b) Calculate the value of $f'(0)$ when f is differentiable and $|f(x)| \leq x^2$ for all $x \in \mathcal{R}$.

[20+10]

3. Suppose a set of $N = \{1,2, \ldots, n\}$ political parties participated in an election; $n \geq 2$. Suppose further that there were a total of V voters, each of whom voted for exactly one party. Each party $i \in N$ received a total of V_i votes, so that $V = \sum_{i=1}^{n} V_i$. Given the vector $(V_1, V_2, \ldots, V_n)$, whose elements are the total number of votes received by the n different parties, define $P_1(V_1, V_2, \ldots, V_n)$ as the probability that two voters drawn at random *with replacement* voted for different parties and define $P_2(V_1, V_2, \ldots, V_n)$ as the probability that two voters drawn at random *without replacement* voted for different parties. Answer the following questions.

(a) Derive the ratio $\frac{P_2}{P_1}$ as a function of V alone.

(b) Consider the special case where $V_i = \frac{V}{n}$ for all $i \in N$. For this case, find the probabilities P_1 and P_2.

[25+5]

Group B

4. Consider an agent living for two periods, 1 and 2. The agent maximizes lifetime utility, given by:

$$U\,(C_1) + \frac{1}{(1+\rho)}\,U\,(C_2),$$

where $\rho > 0$ captures the time preference, while C_1 and C_2 are the agent's consumption in period 1 and period 2, respectively. The agent supplies one unit of labor inelastically in period 1, earning a wage w. A portion of this wage is consumed in period 1 and rest is saved (denoted s). In period 2 the agent does not work, but receives interest income on the savings. Principal plus the interest income on savings goes to finance period 2 consumption. Thus, $C_1 + s = w$ and $C_2 = (1+r)s$, where r is the rate of interest. Assume that the per period utility function can be represented by (and only by) any positive linear transformation of the form $U(C) = \frac{C^{1-\theta}-1}{1-\theta}$, where $0 < \theta < 1$.

(a) Demonstrate, deriving your claim, how optimal savings, s, would respond to changes in r .

(b) Now suppose, initially, $r = \rho$. What happens to optimal savings, s, if r and ρ increase by the same amount (so that the condition $r = \rho$ continues to hold)?

[20+10]

5. A profit maximizing monopolist produces a good with the cost function $C(x) = cx, c > 0$, where x is the level of output, $x \geq 0$. It sells its entire output to a single consumer with the following utility function:

$$u(y) = \theta\sqrt{y} - T(y);$$

where y is the amount of the good purchased by the consumer and T is the payment made by the consumer to the monopolist to purchase the output; $0 \leq y \leq x$; $\theta > 0$. Suppose

$$T(y) = py + t;$$

where $p \geq 0, t \geq 0$ if $y > 0$, and $T(0) = 0$. Thus, in order to purchase any positive amount of the good, the consumer may have to pay a lump-sum amount t, or a per unit price p, or both.

(a) Find the profit of the monopolist when it can choose any non-negative combination of t and p.

(b) Find the profit of the monopolist when it can choose any non-negative p, but is forced to set $t = 0$. Calculate how this profit relates to the profit derived in part (a) and explain your result.

(c) Calculate when social surplus is higher, explaining your result.

(d) Calculate when consumer's surplus is higher, explaining your result.

[10+10+5+5]

6. Answer the following questions.

(a) Let the input demand functions of a profit-maximizing competitive firm operating at unit level of output be given by:

$$x_1 = 1 + 3w_1^{-(1/2)} w_2^{a} \text{ and } x_2 = 1 + b w_1^{(1/2)} w_2^{c} ;$$

where w_1 and w_2 are input prices. Find the values of the parameters a, b and c.

(b) Check whether the following data, summarizing the observed input-output choices of a competitive firm under three different output-input price situations, are consistent with the hypothesis of profit maximization by that firm.

	P	w	q	x
Observation 1	50	20	20	25
Observation 2	45	15	24	36
Observation 3	40	20	16	16

Here p and w denote output price and input price, respectively, while q and x denote, respectively, the units of output supplied and input demanded by the firm. Each of the three rows specifies an observation of the output-input price configuration and the output-input choice of the firm under that particular price configuration.

(c) Suppose, for the production function $f(x_1, x_2)$, the cost function of a competitive firm is $c(q; w) = w_1^{a} w_2^{1-a} q$, where $w = (w_1, w_2)$ is the input price vector and q is the level of output; $\alpha \in (0,1)$. Derive the conditional input demand functions and the production function of the firm.

[10+10+(5 +5)]

Group C

7. Consider a world economy consisting of Home (*H*) and Foreign (*F*). Each of these countries produces a single good that is both consumed domestically and exported. Let Foreign output be the numeraire and let p be the relative price of the *H* produced good. Assume full employment in both countries, so that *H* produces a fixed output Y and *F* produces a fixed output Y^*. Let E be the Home expenditure in terms of its own good and let E^* be the Foreign expenditure measured in terms of the foreign good. We will treat E as a parameter of the model, while E^* is endogenous. Assume that consumers have Cobb-Douglas utility functions with fixed expenditure shares. Let α be the share of expenditure of Home consumers on the Foreign produced good and let α^* be the share of expenditure of Foreign consumers on the Home produced good. Assume further $1-\alpha > \alpha^*$ (i.e., the expenditure share of Home consumers on the Home produced good is greater than the expenditure share of Foreign consumers on the Home produced good). World income equals world expenditure, and goods markets clear.

Now, suppose E falls.

(a) What will happen to p?

(b) What will happen to the trade balance of Home, denominated in units of the Foreign good (i.e., to $p(Y-E)$)?

Prove your claims.

[20+10]

8. Consider the Solow growth model with constant average propensity to save s, labor supply growth rate n, no technological progress and zero rate of depreciation. Let v denote the capital-output ratio.

   (a) Prove that, at the steady state, $\frac{s}{v} = n$.

   (b) Now suppose that, in some initial situation, $\frac{s}{v} > n$. Explain how market forces will operate to restore, over time, the equality $\frac{s}{v} = n$.

   (c) In the process of adjustment in (b), in which direction will the real wage and real rental on capital change? Explain.

[8+16+6]

9. Answer the following questions.

   (a) Using a simple Keynesian model of income determination, derive and explain the conditions under which a rise in the marginal propensity to save will reduce aggregate savings in the economy.

   (b) Using a model of aggregate demand and aggregate supply, explain how an increase in fuel prices would impact aggregate output, employment and the price level.

[15+15]