Sample PQB 2020

PART I (STATISTICS / MATHEMATICS STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- S1 & S2.**

GROUP S1: Statistics

1. **(a)** The registrar of a university keeps an alphabetical list of all students and labels them as $1, 2, \cdots$. The registrar wants to conduct a survey to get some idea about the socio-economic background of the students. Suppose there are 10,000 students in the current year. It was suggested that a random number from 1 to 100 be chosen and the corresponding student be included in the sample. Subsequently every 100th student, starting the count at the selected student's label, is chosen for the sample. Is this a simple random sample? Justify your answer.

[6]

(b) A measuring instrument is being used to obtain n independent measurements $X_i, i = 1, 2, \cdots, n$ of a physical, unknown constant μ. Suppose that the measuring instrument is known to be biased to the positive side by 0.1 units.

(i) Write a model to express the random variable X_i in terms of the bias, the unknown constant μ and the measurement error.

(ii) Obtain the MLE of the variance of error assuming that the observed measurements follow normal distribution.

[2+4]

(c) Let $(X,\ Y) \sim N_2(\mu_1,\ \mu_2, \sigma_1^2, \sigma_2^2, \rho)$. The parameters are unknown. Derive a procedure for testing the following hypothesis:

$$H_0: \frac{\sigma_1^2}{\sigma_2^2} = \theta_0^2 \quad \text{vs} \quad H_1: \frac{\sigma_1^2}{\sigma_2^2} \neq \theta_0^2.$$

[8]

2. Consider the following linear model under the usual Gauss-Markoff set-up.

$$E(y_1) = E(y_3) = \beta_0 + \beta_1,\ E(y_2) = \beta_0 + \beta_2.$$

(a) Obtain least square estimators of the parameters β_0, β_1 and β_2. Are these unique? Are these unbiased? [Justify the preceding two answers.]

[6+2+4]

(b) Determine the condition of estimability of the parametric function $\sum_{i=0}^{2} l_i\, \beta_i$, where l_0, l_1 and l_2 are known constants.

[5]

(c) Obtain the BLUE of $(3\beta_0 + \beta_1 + 2\beta_2)$, if it exists, and also find its variance.

[2+1]

3. Some legal experts believe that the verdict of death penalty in cases of homicide (murder) in a country depends on the race (white / black) of the victim and the defendant (the person accused of the murder). A detailed study covering 326 cases of homicides was carried out and the following were observed:

> Out of 151 cases where both defendant and victim were white, 19 were awarded death penalty. Out of 9 cases where defendant was white and victim black, none was awarded death penalty. Out of 63 cases where defendant was black and the victim white, 11 was awarded death penalty. Finally, out of 103 cases where both defendant and victim were black, 6 were awarded death penalty.

(a) What random variable was being studied? Write the probable distribution of the random variable and list its parameters.

[2 + 2 + 2]

(b) Suppose you are checking the belief regarding discrimination towards awarding death penalty with respect to the race of the defendants (accused) and the victims.

(i) Formulate this problem as testing of statistical hypotheses, stating the null and the alternative hypotheses.

(ii) Are the hypotheses simple or composite? Justify.

[4+2]

(c) Present the entire data on awarding death penalty in tabular format showing race of both the defendant as well as the victim. Use this table to estimate the parameters in the proposed hypotheses. Suggest a way to present this data graphically.

[3 + 3 + 2]

4. Let $X_1, \cdots, X_n$ be a random sample from a distribution with probability density function

$$f(x) = \begin{cases} \frac{x}{\theta^2} e^{\frac{-x}{\theta}}, & x > 0 \\ 0, & \text{otherwise,} \end{cases}$$

with $\theta > 0$.

(a) Find the minimal sufficient statistic of θ. **[5]**

(b) Find the MVUE of θ. **[5]**

(c) Derive a test for $H_0: \theta = \theta_0$ vs $H_1: \theta > \theta_0$. Find the power of the test.

[8+2]

5. **(a)** Let $X_1, \cdots, X_n$ be a random sample from a population with mean μ and variance σ^2. Show that

$$T_n = \frac{1}{2n(n-1)} \sum_i \sum_j \left(X_i - X_j\right)^2$$

is an unbiased estimator of σ^2.

[7]

(b) Consider a population having probability density function

$$f(x) = \begin{cases} (1-\theta)x^{-\theta}, & 0 < x < 1 \\ 0, & \text{otherwise,} \end{cases}$$

with $0 < \theta < 1$.

(i) Based on a random sample $X_1, \cdots, X_n$, find the MLE of θ. Find the Fisher information about θ.

(ii) Let $x_1 = 0.4$, $x_2 = 0.1$, $x_3 = 0.6$ and $x_4 = 0.5$ be a random sample of size 4. Derive an estimate of θ by the method of moments.

[(4+4)+5]

GROUP S2: Probability

6. **(a)** A student takes a multiple-choice test consisting of two questions. The first one has three options and the second one has five. Exactly one option is correct for each of the two questions. The student chooses at random, one option from each of the two questions. Let X be the number of right answers. Find $E(X)$ and $Var(X)$.

[4+6]

(b) Consider the schooling system in a locality, where 15% of the children come from low-income families. Children from low-income families in the locality graduate from college only 20% of the time. Children not from low-income families have 40% chance of graduating from college. One person, picked randomly from the locality, is found to have graduated from a college. What is the probability that this person comes from a low-income family?

[10]

7. Suppose $\{N(t),\ t \geq 0\}$ is a Possion process with rate $\lambda > 0$, where $N(t)$ denotes the number of occurrences of an event $\mathcal{E}$ in $[0, t)$.

(a) Let T denote the time until the first occurrence of the event $\mathcal{E}$. Find the distribution of T.

[5]

(b) Find $P(N(s) = k|N(t) = n)$ for $k\epsilon\{0, 1, \cdots, n\}$ and $n\epsilon\{0, 1, 2, \cdots\}$ for $0 \leq s < t$.

[8]

(c) Suppose that a fair coin is tossed until a head appears. Given that the coin is tossed four times and no head is obtained in these four tosses, what is the probability that the coin must be tossed at least two more times to obtain the first head?

[7]

8. **(a)** Suppose X and Y are jointly distributed with probability density function

$$f(x, y) = \begin{cases} \frac{12}{7}(x^2 + xy), & 0 \leq x \leq 1, \quad 0 \leq y \leq 1 \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of $U = \max\{X, Y\}$. **[10]**

(b) Suppose (X, Y) have a bivariate distribution. Given that $E(X) = 0$, $Var\,(X) = 1$, $E(Y|X) = 2 + 0.5X$ and $Var(Y|X) = 0.75$, find the correlation coefficient between X and Y.

[10]

9. Consider the function defined as follows:

$$F(x) = \begin{cases} 0, & x < 0 \\ 0.3, & 0 \leq x < 2 \\ 0.3 + 0.2x, & 2 \leq x < 3 \\ 1, & x \geq 3. \end{cases}$$

(a) Draw a rough sketch of $F(x)$ on your answer-sheet and argue that it is the distribution function of some random variable X.

[6]

(b) Find $E(X)$. **[10]**

(c) What is the third quartile of X? **[4]**

10. **(a)** Let X be a random variable with the following probability density function:

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 1, & 1 \le x \le 1.5 \\ 0, & \text{otherwise.} \end{cases}$$

Find $E(X|X \ge 0.5)$.

[10]

(b) Let X be a gamma random variable with shape parameter 3 and scale parameter 1. Let Y be exponentially distributed with mean 1. Suppose that X and Y are independently distributed. A new random variable Z is defined as follows:

$$Z = \begin{cases} 1, & \text{if } Y \ge X \\ 0, & \text{otherwise.} \end{cases}$$

Find the distribution of Z.

[10]

PART II (ENGINEERING STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- E1 & E2.**

GROUP E1: Mathematics

1. (a) You are given an acute angled triangle ABC. Draw the perpendiculars AD and BE on the sides BC and CA respectively. Show that CD.BC = CE.AC.

[6]

(b) Find the number of divisors and the sum of the divisors of 5400.

[8]

(c) What is the value of k such that coefficient of x^k is largest in the expansion of $\left(1+\frac{3}{4}x\right)^{15}$?

[6]

2. (a) Consider the curve $y = x^3 + 2$. Find the equation of its tangent passing through the origin. Also, evaluate the area of the region enclosed by this tangent and the curve.

[6+8]

(b) Find the number of ways of choosing three numbers a, b and c from the integers 1, 2, …, 201 such that these three numbers form an A.P.

[6]

3. (a) Determine the radius of convergence of the series

$$\sum_{n=0}^{\infty}\left(\frac{26}{11}\right)^n \frac{n^3x^{3n}}{n^4+1}.$$

[10]

(b) Suppose that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3}=1$$

for some constants a and b. Show that $a=-\frac{5}{2}$ and $b=-\frac{3}{2}$.

[10]

4. **(a)** For $n \geq 2$, consider the following square matrix of order (n-1):

$$\begin{pmatrix} 3 & 1 & 1 & 1 & & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & & 1 \\ 1 & 1 & 1 & 6 & & 1 \\ & \vdots & & & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{pmatrix}$$

Find its determinant using only elementary row operations and denote it by A_n. Hence or otherwise, check whether the sequence $\{{}^{A_n}/_{n!}\}_{n\geq 2}$ is bounded.

[12+3]

(b) Solve the inequality $\frac{x+3}{x^2-5x+6} \leq 0$ for x.

[5]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. **(a)** A particle, starting from rest, moves in a straight line, whose acceleration is given by the equation $a = 10 - 0.006S^2$ where a = acceleration in m-s^{-2}, S = distance traversed in m. Determine

(i) the velocity of the particle, when it has travelled 50 m,
(ii) the distance travelled by the particle when it comes to rest.

[3+3]

(b) In a certain weight lifting machine, a weight of 16000 N is lifted by an effort of 400 N while the weight moves up by 150 mm. The point of application of effort moves up by 9000 mm. Find

- **(i)** the mechanical advantage
- **(ii)** the velocity ratio
- **(iii)** the efficiency of the machine
- **(iv)** whether the machine is reversible or not with proper explanation.

[2+2+2+2]

(c) A man wishes to move a wooden block a distance of 2 m to the right with the least amount of work. If the block weighs 1000 N and the coefficient of friction is 0.3, should he slide it as indicated in Figure 1a or should he tip it as indicated in Figure 1b?

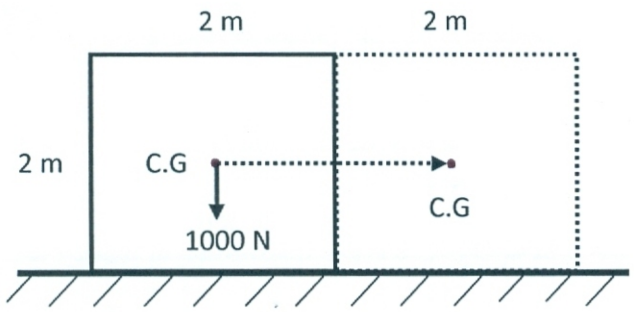

Figure 1a

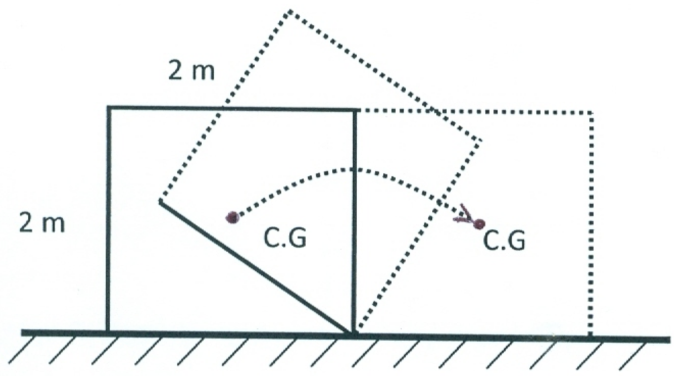

Figure 1b

[6]

6. **(a)** The food compartment of a refrigerator is maintained at 4^0C by removing heat from it at a rate of 360 kJ per minute. If the required power input to the refrigerator is 2 kW, determine

(i) the coefficient of performance of the refrigerator,
(ii) the rate of heat rejection to the room that houses the refrigerator.

[5+5]

(b) A heat engine is used to drive a heat pump. The heat transferred from the heat engine and the heat pump are rejected to the same sink. The efficiency of the heat engine is 27% and the COP of the heat pump is 4. Determine the ratio of the *total heat rejection from the heat engine and the heat pump* to the *heat transferred to the heat engine.*

[10]

7. **(a)** A crate of weight W rests with one end on an inclined plane, the opposite edge being supported by means of a rope as shown in Figure 2. The edge at A is on the verge of slipping down. The angle of friction is 30^0. Show that the tension T on the rope is $W/\sqrt{3}$.

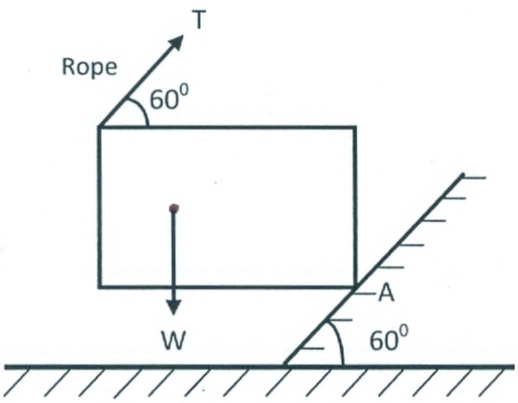

Figure 2

[10]

(b) A stationary mass of gas is compressed without friction from an initial state of 0.3 m^3 and 0.105 MPa to a final state of 0.15 m^3 and 0.105 MPa. There is a transfer of 37.6 kJ of heat from the gas during the process. What is the change in internal energy of the gas during the process?

[10]

Electrical and Electronics Engineering

8. **(a)** An inductive load of 0.8 lagging power factor is fed by a 250 V / 125 V, 1 kVA single phase transformer. The core loss obtained from the no-load test and the copper loss obtained from the short-circuit test of the transformer are 80 W and 120 W respectively.

(i) Determine the full load efficiency of the transformer.

(ii) Write down the condition of maximum efficiency and estimate the ratio of the load currents under maximum efficiency to the full load efficiency. Neglect the no-load current.

[5+(2+3)]

(b) A six-pole, wave-wound DC shunt motor is drawing 11 A line current when connected across a 220 V DC supply. The armature resistance and field resistance of the motor are 4 Ω and 220 Ω respectively.

(i) Determine the back emf developed in the armature.

(ii) Determine the speed of rotation in RPM, if the armature has 400 conductors and the flux per pole is 0.01 Wb.

[5+5]

9. **(a)** Design the circuit for the Boolean function $Y = AB + \overline{A}C$ using NAND gates only.

[4]

(b) Subtract 11110 from 10011 using 1's complement method.

[4]

(c) Consider the circuit shown in Figure 3.

(i) Find the Thevenin equivalent for the network to the left of terminal AB by applying Thevenin's theorem.

(ii) Find the output voltage v_0 using the Thevenin voltage across the terminal AB.

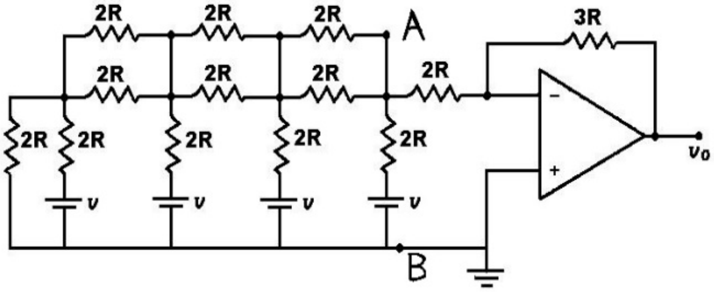

Figure 3

[8+4]

Engineering Drawing

10. (a) Sketch the isometric view of the object whose two views are shown in Figure 4.

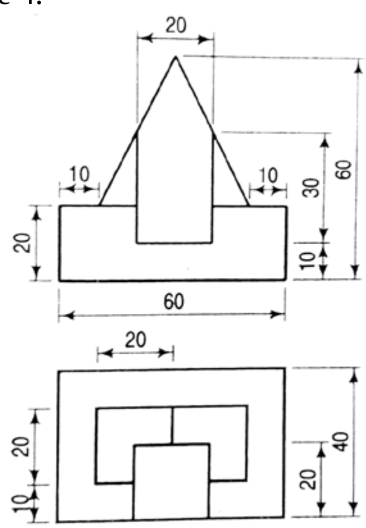

Figure 4

[10]

(b) Sketch the front view and top view of the object shown in Figure 5. Use third angle projection.

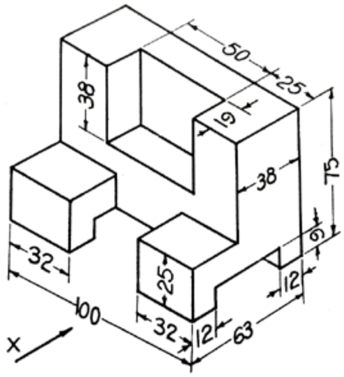

Figure 5

[10]