PART I (STATISTICS / MATHEMATICS STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- S1 & S2.**

GROUP S1: Statistics

1. (a) At a toll plaza, it takes at least α minutes for a car to pay the toll and move on. The actual time (X) varies from car to car. It is further assumed that this time is well represented by an exponential random variable with the following density function.

$$f_X(x) = \begin{cases} \frac{1}{\theta} e^{(\alpha - x)/\theta}, & x \geq \alpha, 0 < \theta < \infty \\ 0 & \text{Otherwise.} \end{cases}$$

The following time values (in minutes) are recorded from 10 cars, selected randomly.

4.1, 3.2, 4.6, 5.3, 3.7, 6.9, 4.3, 3.5, 3.9, 4.2

Derive the maximum likelihood estimators of α and θ, and give the respective estimates.

[10]

(b) Consider the model $y_i = \beta x_i + \epsilon_i$ for $i = 1(1)n$ with $x_i > 0$, where ϵ_i's are independent $N(0, x_i \sigma^2)$ and, β and σ^2 are unknown constants. Suppose that $\hat{\beta}_1$ and $\hat{\beta}_2$ are the least-squares estimators of β obtained by minimizing

$$S_1 = \sum_{i=1}^{n} (y_i - \beta x_i)^2 \quad \text{and} \quad S_2 = \sum_{i=1}^{n} \frac{1}{x_i} (y_i - \beta x_i)^2 ,$$

respectively.

Show that $\text{Var}\left(\hat{\beta}_1\right) \geq \text{Var}\left(\hat{\beta}_2\right)$.

[10]

2. The effect of five different additives on the yield of a chemical process is to be studied. Each run of the experiment requires a raw material which is supplied in batches. Each batch of material is only large enough to permit five runs to be made. Also due to time and infrastructure constraints, only five runs can be made in a day. There is no interaction between the factors.

(a) Plan the experiment using any of the standard complete block designs known to you, in not more than twenty five runs, to compare the effects of the additives, and give the layout of the experiment. Also, write down the associated linear model and state the assumptions, if any. How do you honour the three basic principles of experimentation?

[5+3+5 = 13]

(b) If the experiment is to be conducted at two different locations, adopting the same design proposed by you in (a), then outline a method of combined analysis of the data collected from the two places, assuming error variances are the same at both the places. [No derivation is necessary]

[7]

3. (a) A batch of chemicals may pass or fail a particular test carried out after production. A failed batch is corrected and it undergoes the same test for the second time, where the batch may pass or fail. Notice that there are three possible outcomes for a batch – pass, fail-pass and fail-fail. Suppose we have collected test data for n batches.

(i) Find the maximum likelihood estimators of the probabilities of the three possible outcomes.

(ii) Test the hypothesis that the probability of failing in the first test is same as the conditional probability of failing in the second test against the alternative that these are not same.

[5+8 = 13]

(b) Consider the random variable X having probability mass function

$$P[X=x] = \begin{cases} \theta + (1-\theta)e^{-\lambda}, & x = 0 \\ \dfrac{(1-\theta)e^{-\lambda}\lambda^x}{x!}, & x = 1,2,\cdots, \end{cases}$$

$\lambda > 0$ and $0 < \theta < 1$.

In a random sample of size n from the above distribution, suppose that n_1 of the observations are 0 and the rest are non-zero. For a known λ, obtain an unbiased estimator of θ based on the given information.

[7]

4. (a) Suppose we want to estimate the average number of school going children per household in a city. We have the complete list of schools. We choose a school at random and talk to the children in the school to find the number of school going children in their respective households. Assume that the children have given the number correctly and households are well defined.

(i) What are the study and target populations?

(ii) Do you think the sample average of the number of school going children per household would be an unbiased estimator of the average of the target population? Justify.

[2+5 = 7]

(b) A trial involves randomly choosing a number from the interval (0, 1) following some fixed and unknown scheme. Based on $n = 80$ independent trials, the frequencies observed in different subintervals are

Subinterval	(0, 1/4]	(1/4, 1/2]	(1/2, 3/4]	(3/4, 1)
Frequency	6	18	20	36

Consider the null hypothesis (H_0) that the underlying scheme of choosing each number is governed by the pdf $f(x) = 2x$

for $0 < x < 1$. Carry out a test to check whether H_0 can be accepted at 5% (approximate) level of significance.

[$\chi^2_{0.025,4}$= 11.143, $\chi^2_{0.05,4}$= 9.488, $\chi^2_{0.025,3}$= 9.348, $\chi^2_{0.05,3}$= 7.815, where $\chi^2_{\alpha,m}$ is the upper α-point of χ^2 distribution with m degrees of freedom.]

[13]

5. Let $X_1, X_2, \ldots, X_n$ be a random sample from $N(0, \theta)$, $\theta > 0$.

(a) Derive a sufficient statistic for θ.

(b) Find the Cramer-Rao lower bound for the variance of an unbiased estimator of θ. Is the unbiased estimator efficient? – Justify your answer.

(c) Derive a 95% confidence interval for θ.

(d) Derive an α-level most powerful test for $H_0: \theta = \theta_0$ versus $H_1: \theta > \theta_0$.

[3+(4+4)+5+4 = 20]

GROUP S2: Probability

6. Suppose the bivariate random variable (X, Y) is uniformly distributed over the region bounded below by $y = x - 1$ for $1 \leq x \leq 2$ and $y = 3 - x$ for $2 \leq x \leq 3$ and, bounded above by $y = x$ for $1 \leq x \leq 2$ and $y = 4 - x$ for $2 \leq x \leq 3$.

(a) Find the joint probability density function $f_{X,Y}(x, y)$.

(b) Find the marginal probability density functions $f_X(x)$ and $f_Y(y)$ of X and Y respectively.

(c) Are X and Y independent?

(d) Find $E(Y|X = x)$.

[7+(3+5)+1+4 = 20]

7. (a) Let X_1, X_2, X_3 be independent and identically distributed continuous random variables. Compute $P(X_1 > X_3 | X_1 > X_2)$.

[10]

(b) If X is a non-negative integer-valued random variable, then show that $E(X) = \sum_{x=1}^{\infty} P(X \geq x)$.

[10]

8. (a) Suppose $(X_1, X_2, X_3)^T \sim N_3(\mu, \Sigma)$, where

$$\mu = \begin{bmatrix} -3 \\ 1 \\ 4 \end{bmatrix} \text{ and } \Sigma = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 5 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(i) Find the conditional variance-covariance matrix of (X_1, X_2) given $X_3 = x_3$.

(ii) Find the partial correlation coefficient between X_1 and X_2 for fixed X_3.

[8+2 = 10]

(b) Let X be a random variable with probability density function

$$f(x) = \begin{cases} 1/3 & \text{if } -1 \leq x \leq 2 \\ 0 & \text{otherwise.} \end{cases}$$

(i) Find the probability density function of $Y = X^2$.

(ii) Find the 95th percentile point of Y.

[8+2 = 10]

9. (a) Consider a Markov chain $\{X_n, n \geq 0\}$ with state space $S = \{1, 2\}$ and the transition probability matrix

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix}.$$

(i) Find the distribution of X_2, given the probability distribution of X_0 as $(1/2, 1/2)^T$.

(ii) Find $P(X_8 = 2, X_7 = 1, X_5 = 2 \mid X_3 = 2, X_2 = 1)$.

(iii) What are the steady-state probabilities?

[5+5+2 = 12]

(b) Two dice are being rolled together repeatedly, where the outcome of interest is the sum of their faces as shown up. What is the probability that *Seven* occurs before *Eight*?

[8]

10. (a) Suppose that $X_1, X_2, \ldots, X_n$ are independent and identically distributed $U(0, \theta)$ variables. Let $Y_n = \max\{X_1, \ldots, X_n\}$. Find the limiting distribution of $Z_n = n(\theta - Y_n)$.

[10]

(b) Customers arrive at a service station according to a Poisson process with rate $\lambda = 10$ per hour.

(i) Suppose that eight customers arrive during a given hour, what is the probability that at most two customers will arrive over the following hour?

(ii) Suppose that a customer arrived during a 30-minute period, what is the probability that she arrived during the first 15 minutes of the period considered?

[5+5 = 10]

PART II (ENGINEERING STREAM)

ATTENTION: Answer a total of **SIX** questions taking **at least TWO** from each Group **- E1 & E2.**

GROUP E1: Mathematics

1. (a) Find the area of the closed figure bounded by $x = -1$, $y = 0$, $y = x^2 + x + 1$ and the tangent to the curve $y = x^2 + x + 1$ at (1, 3).

[8]

(b) A conical tent of a given capacity has to be constructed. Find the ratio of the height to the radius of the base for the minimum amount of the canvas required for the tent.

[12]

2. (a) Two complex numbers Z_1 and Z_2 lie on a circle with center at the origin. Further, the tangents to the circle at Z_1 and Z_2 intersect at Z_3. Show that Z_3 can be expressed as

$$Z_3 = \frac{2\, Z_1 Z_2}{Z_1 + Z_2}.$$

[10]

(b) A periodic function $f(x)$ with period 2π is defined as

$$f(x) = x^2, \qquad -\pi \le x < \pi.$$

(i) Sketch the graph of the function between -5π and 5π.

(ii) Determine its Fourier coefficients and Fourier series.

[2+8 = 10]

3. (a) A total of *N* badminton players participated in a recently concluded round robin singles tournament where each player played every other player exactly once. Each match produced either a winner or a loser. Let W_i be the number of matches won by the *i*-th player and L_i be the number of matches lost by the *i*-th player in the tournament. Show that $\sum_{i=1}^{N} W_i^2 = \sum_{i=1}^{N} L_i^2$.

[6]

(b) Let $u = \boldsymbol{x^T A x}$ where $\boldsymbol{x} = (x_1, x_2, x_3)^T$ is a 3 × 1 vector and $\boldsymbol{A} = \begin{pmatrix} 1 & 3 & -2 \\ -3 & 2 & 0 \\ 2 & 0 & 3 \end{pmatrix}$. Find the vector $\boldsymbol{x}$ that minimizes u subject to the constraints $x_1 + x_2 = 2$ and $x_2 + x_3 = 3$.

[7]

(c) The four points A$(0,0)$, B$(1,0)$, C$(3,0)$ and D$(5,0)$ are given. Find the locus of the point P satisfying $\angle APB = \angle CPD$.

[7]

4. (a) Two concentric circles of radii R and r, $R > r$, are shown in Figure 1. From a point P, on the smaller circle, a straight line is drawn that intersects the larger circle at B and C. The perpendicular to BC at P intersects the smaller circle at A.

Show that $PA^2 + PB^2 + PC^2 = 2(R^2 + r^2)$.

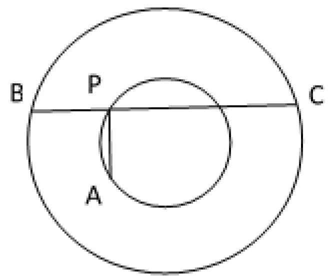

Figure 1

[10]

(b) A sequence of numbers a_n, $n = 1, 2, \cdots$ is defined as follows

$$a_1 = \frac{1}{2} \text{ and } a_n = \left(\frac{2n-3}{2n}\right) a_{n-1} \text{ for } n \geq 2.$$

Prove that $\sum_{k=1}^{n} a_k < 1$ for all $n \geq 1$.

[10]

GROUP E2: Engineering & Technology

Engineering Mechanics and Thermodynamics

5. (a) A car, starting from rest, is uniformly accelerated. The acceleration at any instant is $\frac{5}{(V+2)}$ m-s^{-2}, where V is the velocity of the car in m-s^{-1} at the instant. Find the distance in which the car will attain a velocity of 36 km-h^{-1}.

[6]

(b) An axial pool of 50 kN is suddenly applied to a steel rod of length 2 m and cross sectional area of 1000 mm^2. Calculate the strain energy that can be absorbed if Young's modulus of steel is 20×10^4 N-mm^{-2}.

[6]

(c) A bar is hanging freely. The cross section of the bar is A and the length is l. It is rigidly fixed at the upper end. The Young's modulus of the bar is E. Derive an expression of total elongation of the bar due to its own weight.

[8]

6. (a) A reversible heat engine receives 8 kJ of heat from one thermal reservoir at a temperature of 800 K and 7 kJ of heat from another thermal reservoir at a temperature of 700 K. If it rejects heat to a third thermal reservoir at a temperature of 200 K, find the thermal efficiency of the engine.

[10]

(b) A domestic food freezer maintains a temperature of -15^0 C. The ambient air is at 30^0 C. If the heat leaks into the freezer at a continuous rate of 1.75 kW, what is the least power necessary to pump the heat out continuously?

[10]

7. (a) Find the amount of heat required to convert 4 kg of water at 20^0 C to ice at -2^0 C (specific heat of ice = 2.09 kJ kg^{-1}K^{-1}).

[5]

(b) An ideal gas expands from a volume of 6×10^{-3} m^3 to 16×10^{-3} m^3 against constant external pressure of 2.026×10^5 N-m^{-2}. Find the change in enthalpy if the change in internal energy, $\Delta U = 418$ J.

[5]

(c) For the system shown in Figure 2, the masses of the blocks A, B and C are m_1, m_2 and m_3 respectively, where $(m_1 + m_2) < m_3$. Find the tension T_1, T_2 and the common acceleration of the blocks. Assume the pulley to be light and the connecting strings of negligible masses.

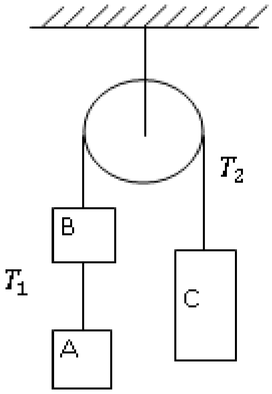

Figure 2

[3+3+4 = 10]

Electrical and Electronics Engineering

8. (a) The number of turns in the low voltage (LV) side and the high voltage (HV) side of a 2 kVA, 50 Hz single phase transformer are 100 and 500 respectively.

   (i) Under rated supply conditions, if the 'voltage per turn' of the LV side is 2.0 V, determine the induced emf in the HV side.

   (ii) Estimate the rated currents in the LV side and the HV side of the above-mentioned transformer.

   (iii) Calculate the percentage voltage regulation of the transformer under a certain loaded condition when the terminal voltage in the HV side is measured to be 940 V.

   [3+(2+2)+3 = 10]

   (b) A 2-pole, separately excited DC generator is required to produce 600 V for an electroplating factory. The total number of conductors in the armature of the generator is 600 and the magnetic flux per pole is 0.02 Wb.

   (i) Estimate the speed in rpm at which the generator should be driven by a prime-mover.

   (ii) If the resistance of each armature conductor is 0.012 Ω, determine the equivalent resistance of the armature circuit. Neglect the brush contact resistances.

   (iii) Calculate the net armature copper loss of the generator if the load current is 10 A. Neglect the brush contact drop.

   [4+4+2 = 10]

9. (a) An amplifier with negative feedback has an overall gain of 100. Open-loop gain increase of 10% is expected owing to production limitations. Determine the value of open-loop gain and feedback fraction β for which closed-loop gain will only increase by 1%.

   [9+3 = 12]

(b) Determine the voltage across the 10 Ω resistor in Figure 3 using the source transformation method.

[5]

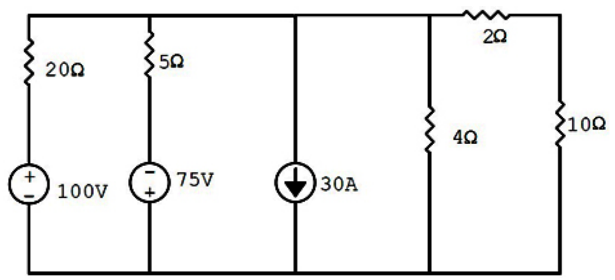

Figure 3

(c) Find the octal equivalent of 879.

[3]

Engineering Drawing

10. (a) The front view and right side view of an object are shown in Figure 4. Draw/sketch the top view and isometric view of the object.

[8]

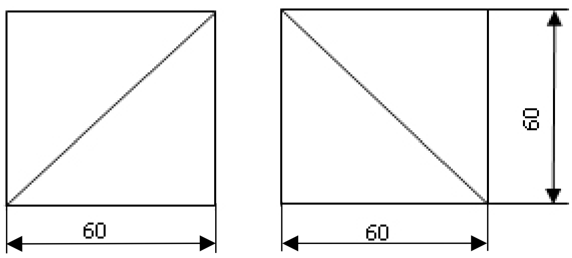

Figure 4

(b) Draw/ sketch the three views of the object shown in Figure 5. Put dimensions on the views.

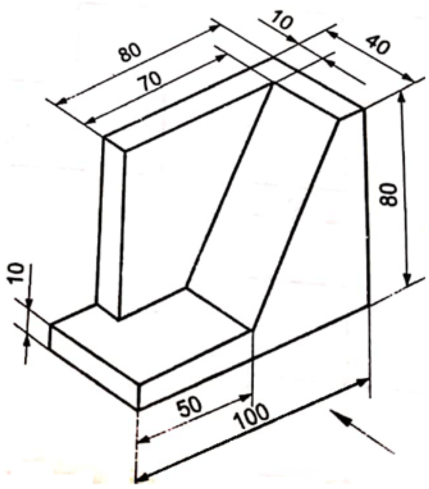

Figure 5

[12]