2014

BOOKLET No. TEST CODE : **UGB**

Afternoon

Answer all questions	**Time : 2 hours**

Write your Registration number, Test Centre, Test Code and the Number of this booklet in the appropriate places on the answer-booklet.

ALL ROUGH WORK MUST BE DONE ON THIS BOOKLET AND/OR ON THE ANSWER BOOKLET. YOU ARE NOT ALLOWED TO USE A CALCULATOR.

Answer to each question should start on a fresh page

STOP! WAIT FOR THE SIGNAL TO START.

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1. A class has 100 students. Let a_i, $1 \le i \le 100$, denote the number of friends the i-th student has in the class. For each $0 \le j \le 99$, let c_j denote the number of students having *at least* j friends. Show that

$$\sum_{i=1}^{100} a_i = \sum_{j=0}^{99} c_j \, .$$

2. It is given that the graph of $y = x^4 + ax^3 + bx^2 + cx + d$ (where a, b, c, d are real) has at least 3 points of intersection with the x-axis. Prove that either there are *exactly* 4 distinct points of intersection, or one of those 3 points of intersection is a local minimum or maximum.

3. Consider a triangle PQR in $\mathbb{R}^2$. Let A be a point lying on $\triangle PQR$ or in the region enclosed by it. Prove that, for any function $f(x, y) = ax + by + c$ on $\mathbb{R}^2$,

$$f(A) \le \max\{f(P), f(Q), f(R)\} \, .$$

4. Let f and g be two non-decreasing twice differentiable functions defined on an interval (a, b) such that for each $x \in (a, b)$, $f''(x) = g(x)$ and $g''(x) = f(x)$. Suppose also that $f(x)g(x)$ is linear in x on (a, b). Show that we must have $f(x) = g(x) = 0$ for all $x \in (a, b)$.

5. Show that the sum of 12 consecutive integers can never be a perfect square. Give an example of 11 consecutive integers whose sum is a perfect square.

6. Let A be the region in the xy-plane given by

$$A = \{(x, y) : x = u + v, y = v, u^2 + v^2 \le 1\} \, .$$

Derive the length of the longest line segment that can be enclosed inside the region A.

7. Let $f : [0, \infty) \to \mathbb{R}$ be a non-decreasing continuous function. Show then that the inequality

$$(z - x) \int_y^z f(u)du \ge (z - y) \int_x^z f(u)du$$

holds for any $0 \le x < y < z$. **[P. T. O]**

8. Consider $n\,(> 1)$ lotus leaves placed around a circle. A frog jumps from one leaf to another in the following manner. It starts from some selected leaf. From there, it skips exactly one leaf in the clockwise direction and jumps to the next one. Then it skips exactly two leaves in the clockwise direction and jumps to the next one. Then it skips three leaves again in the clockwise direction and jumps to the next one, and so on. Notice that the frog may visit the same leaf more than once. Suppose it turns out that if the frog continues this way, then all the leaves are visited by the frog sometime or the other. Show that n *cannot* be odd.