Sample Questions 2020
Test Code PCB (Short Answer Type)

- **The questions are divided into *two* groups, Group A (Computer Science) and Group B (Non Computer Science).**
- **Answer the questions from any one of the groups.**

Group A

Computer Science

C1. Let A be a sorted array containing n distinct integers, such that, for all $1 \leq i < j \leq n$, we have $A[i] < A[j]$. Note that the integers stored in the array A are not necessarily from the set $\{1, \ldots, n\}$. Design an algorithm that outputs an index $i \in \{1, \ldots, n\}$ such that $A[i] = i$, if such an i exists, and outputs -1 otherwise. The worst case running time of the algorithm should be asymptotically better than $O(n)$. Prove the correctness of your algorithm and state its asymptotic time complexity.

C2. Let K_n denote the complete graph on n vertices, with $n \geq 3$, and let u, v, w be three distinct vertices of K_n. Determine the number of distinct paths from u to v that do not contain the vertex w.

C3. When we add a pair of two-bit binary numbers, say ab and cd, we get a number of at most three bits, say pqr. Using standard operators of Boolean algebra, namely AND ($\wedge$), OR ($\vee$) and NOT ($\neg$), derive the Boolean expressions of p, q and r in terms of a, b, c and d.

C4. In a binary tree T, for a node v, the `LEFT-HEIGHT(`v`)` is the length of the longest path from v to any leaf in the left subtree of v. If v has no left child then `LEFT-HEIGHT(`v`)` $= 0$. The `RIGHT-HEIGHT(`v`)` is defined accordingly.

A node v is said to be *properly balanced* if

$$|\texttt{LEFT-HEIGHT(}v\texttt{) - RIGHT-HEIGHT(}v\texttt{)}| \leq 1.$$

Design an efficient algorithm that, given a binary tree, enumerates all the nodes which are *properly balanced.*

C5. Consider a stack machine where the only available workspace is a stack whose elements are unsigned integers. We will denote the configuration of the stack by a sequence. For example $[a, b, c, d]$ represents a stack with a being the top most element and d the bottom most element. The stack machine supports the following instructions:

`PUSH` a	Pushes a to the stack, i.e., if $[x, y, z]$ be the stack, after `PUSH` a, the stack becomes $[a, x, y, z]$.
`DUP`	Duplicates the top, i.e., if $[a, b, c]$ be the stack, after `DUP` the stack becomes $[a, a, b, c]$.
`ADD`	Adds the two topmost elements, removes them and pushes the result, i.e., if $[a, b, c]$ be the stack, after `ADD` it becomes $[a + b, c]$.
`SUB`	Subtracts the two topmost elements, removes them and pushes the absolute value of the result, i.e., if $[a, b, c]$ be the stack, after `SUB` it becomes $[\|a - b\|, c]$.
`SQR`	Computes the square of the topmost element, removes it and pushes the result, i.e., if [a, b, c] be the stack, after `SQR` it becomes $[a^2, b, c]$.
`SFT`	Removes the top most element, right shifts the element by 1 bit, and pushes the result, i.e., if [a, b, c] be the stack, after `SFT` it becomes $[\lfloor a/2 \rfloor, b, c]$.
`REV`	Reverses the order of the three topmost elements in the stack, i.e., if $[a, b, c, z]$ be the stack, after REV the configuration becomes $[c, b, a, z]$ (if the stack contains less than 3 elements, REV is undefined).

Computation starts with an empty stack and after a sequence of operations the top most element is considered as the final result. For example, to compute an expression $((x + y)^2 + \lfloor z/2 \rfloor)^2$ the following sequence of instructions may be used:
`PUSH` x; `PUSH` y; `PUSH` z; `SFT`; `REV`; `ADD`; `SQR`; `ADD`; `SQR`

Given two unsigned integers a and b, write a sequence of instructions to compute the product ab. The instruction sequence should start with:

`PUSH` a; `PUSH` b; ...

with no further `PUSH` operations allowed. After the whole sequence of instructions executes, the top of the stack should contain ab.

C6. Consider the alphabet $\Sigma = \{0, 1, 2, ..., 9, \#\}$, and the language of strings of the form $x\#y\#z$, where x, y and z are strings of digit such that when viewed as numbers, satisfy the equation $x + y = z$. For example, the string 123#45#168 is in this language because $123 + 45 = 168$. Is this language regular? Justify your answer.

C7. Recall that in go-back-N protocol, the transmitting window size is N and the receiver window size is 1. Consider a pipelined, reliable transport protocol that uses go-back-N with cumulative acknowledgment. Assume that the timeouts trigger retransmissions (but note that duplicate acknowledgments do not). Further assume that the receiver does not maintain any receive buffer, the one-way delay between the sender and receiver is 50 ms, and every packet is 10,000 bits long. Suppose that the sender must be able to send at a steady rate of 1 Gb/s (gigabit per second) under ideal conditions.

(a) What should be the window size to allow the steady rate mentioned above?

(b) Suppose that the expected number of packets lost per 100,000 packets is 1. If the sender uses a timeout of 500 ms and a window size of 20,000 packets, calculate the expected gap between two timeouts, when

(i) The bottleneck link rate is 1 Gb/s.

(ii) The bottleneck link rate is 2 Gb/s.

C8. Let us assume that a disk scheduling algorithm is applied on a storage disk to access several cylinders (numbered as $0, 1, \ldots, n$ arranged in ascending order) for some I/O operations. These are received in the order $< q, r, s, t >$.

If the disk head starts from the cylinder p, then state the relations among p, q, r, s, and t with proper justifications such that both the scheduling algorithms `SSTF` and `SCAN` perform similarly in terms of the total head movements required.

Note that, in `SSTF` (Shortest Seek Time First) the disk head moves to the nearest cylinder among the unaccessed ones at any point. In `SCAN`, first the disk head accesses all the cylinders while moving toward cylinder 0 and then the disk head moves toward the other end.

C9. Consider a byte addressable memory with 16 bit addresses and a 2-way set associative L1 cache of size 8 kB (kilobyte). Each cache line is 4 words long. A process sequentially accesses the following memory addresses:

`0x1000, 0x1004, 0x1010, 0x11C0, 0x2000, 0x3000, 0x1006, 0x2001`

Assuming the L1 cache is initially empty and the LRU (least recently used) page replacement policy is used, indicate whether the cache access will result in a hit or a miss for each of the above addresses.

C10. Let R be a relation with functional dependencies $\mathcal{F}$. For any subset of attributes $X \subseteq R$, the closure of X is defined as the set

$$X^+ = \{A \in R \mid X \to A \text{ holds with respect to } \mathcal{F}\}.$$

For two non-empty attribute sets Y and Z in R, prove or disprove each of the following statements:

(a) $(Y^+Z)^+ = (YZ)^+$

(b) $(YZ)^+ = Y^+Z^+$

C11. Consider an array of length n consisting only of positive and negative integers. Design an algorithm to rearrange the array so that all the negative integers appear before all the positive integers, using $O(n)$ time and only constant amount of extra space.

C12. You can climb up a staircase of n stairs by taking steps of one or two stairs at a time.

(a) Formulate a recurrence relation for counting a_n, the number of distinct ways in which you can climb up the staircase.

(b) Mention the boundary conditions for your recurrence relation.

(c) Find a closed form expression for a_n by solving your recurrence.

C13. An n-variable Boolean function $f : \{0, 1\}^n \to \{0, 1\}$ is called symmetric if its value depends only on the number of 1's in the input. Let σ_n denote the number of such functions.

(a) Calculate the value of σ_4.

(b) Derive an expression for σ_n in terms of n.

C14. Let the valid moves along a staircase be U (one step *up*) and D (one step *down*). For example, the string $s = UUDU$ represents the sequence of moves as two steps up, then one step down, and then again one step up. Suppose a person is initially at the base of the staircase.

A string denoting a sequence of steps that takes the person below the base is invalid. For example, the sequence $UUDDDU$ is invalid. Let L be the language defined by the set of valid strings which represent scenarios in which the person never returns to the base of the staircase after the final step.

(a) Show that L is not regular.

(b) Write a context free grammar for accepting L.

C15. Consider a max-heap of n distinct integers, $n \geq 4$, stored in an array $\mathcal{A}[1 \ldots n]$. The *second minimum* of $\mathcal{A}$ is the integer that is less than all integers in $\mathcal{A}$ except the minimum of $\mathcal{A}$. Find all possible array indices of $\mathcal{A}$ in which the second minimum can occur. Justify your answer.

C16. The following function computes an array `SPF`, where, for any integer `1 < i < 1000`, `SPF[i]` is the *smallest prime factor* of `i`. For example, `SPF[6]` is 2, and `SPF[11]` is 11.

There are five missing parts in the following code, commented as `/* Blank */`. For each of them, copy the entire line with the comment and fill the blank appropriately in your answer sheet.

```
int SPF[1000];

void findSPF() {
  SPF[1] = 1;

  // Initializing SPF of every number to be itself
  for (int i = 2; i < 1000; i++) {
    _____; /* Blank 1 */
  }

  // SPF of every even number is 2
  for (int i = 4; i < 1000; i += 2) {
    SPF[i] = _____; /* Blank 2 */
  }

  // For odd numbers, updating the SPFs of their multiples
  for (int i = _____; i * i < 1000; i++) { /* Blank 3 */
    if (SPF[i] == i) { // No smaller factor of i found yet
      for (int j = _____; j < 1000; j+= i) { /* Blank 4 */
        if (SPF[j] == j) {
           SPF[j] = _____; /* Blank 5 */
        }
```

```
            }
          }
        }
      }
```

C17. A context switch from a process P_{old} to a process P_{new} consists of the following steps:

- Step I: saving the context of P_{old};
- Step II: running the scheduling algorithm to pick P_{new};
- Step III: restoring the saved context of P_{new}.

Suppose Steps I and III together take T_0 units of time. The scheduling algorithm takes nT_1 units of time, where n is the number of ready-to-run processes. The scheduling policy is round-robin with a time slice of 10ms. Compute the CPU utilization for the following scenario: k processes become ready at almost the same instant in the order $P_1, P_2, \ldots, P_k$; each process requires exactly one CPU burst of 20ms and no I/O burst.

C18. Consider a 5-stage instruction pipeline. The stages and the corresponding stage delays are given below.

Instruction	**Stage delay**
Fetch instruction (FI)	3 ns
Decode instruction (DI)	4 ns
Fetch operand (FO)	7 ns
Execute instruction (EI)	10 ns
Write result (WR)	7 ns

Assume that there is no delay between two consecutive stages. Consider a processor with a branch prediction mechanism by which it is always able to correctly predict the direction of the branch at the FI stage itself, without executing the branch instruction. A program consisting of a sequence of 10 instructions I1, I2, . . . , I10, is executed in the pipeline, where the 5^{th} instruction (I5) is the only branch instruction and its branch target is the 8^{th} instruction (I8).

(a) Draw the pipeline diagram over time showing how the instructions I1, I2, . . . , I10 flow through the pipeline stages in this processor.

(b) Calculate the time (in ns) needed to execute the program.

C19. The data link layer uses a fixed-size sliding window protocol, where the window size for the connection is equal to *twice* the bandwidth-delay product of the network path. Consider the following three scenarios, in each of which only the given parameter changes as specified (no other parameters change). For each scenario, explain whether the throughput (not utilization) of the connection increases, decreases, remains the same, or cannot be determined:

(a) the packet loss rate L decreases to $L/3$;

(b) the minimum value of the round trip time R increases to $1.8R$;

(c) the window size W decreases to $W/3$.

C20. Consider two $n \times 1$ vectors $\mathbf{u}$ and $\mathbf{v}$, stored as tables `U(ind,val)` and `V(ind,val)` with the same schema. A row (i, u_i) of table `U` specifies that the i-th element of vector $\mathbf{u}$ has value u_i (similarly for $\mathbf{v}$, respectively). Only the non-zero entries of the vectors are stored in the corresponding tables. For example, if the vector $\mathbf{u}$ equals $(0, 1, 3, 0, 2, 0)$, then it is represented in table `U` as:

ind	val
2	1
3	3
5	2

Write a relational algebra expression or an SQL query to compute the sum $\mathbf{u} + \mathbf{v}$ of the two vectors $\mathbf{u}$ and $\mathbf{v}$. Explain your solution.

Group B

Non Computer Science

NC1. Let $\{a_n\}$ be a decreasing sequence such that $\sum_{n=1}^{\infty} a_n$ is convergent. Prove that the sequence $\{na_n\}$ goes to zero as $n \to \infty$.

NC2. Consider an $n \times n$ matrix $A = I_n - \alpha\alpha^T$, where I_n is the identity matrix of order n and α is an $n \times 1$ column vector such that $\alpha^T\alpha = 1$. Prove that $A^2 = A$.

NC3. Let A and B be two invertible real matrices of order n. Show that $\det\,(xA + (1-x)B) = 0$ has finitely many solutions for x.

NC4. Show that for every $\theta \in (0, \frac{\pi}{2})$, there exists a unique real number x_θ such that

$$(\sin\theta)^{x_\theta} + (\cos\theta)^{x_\theta} = \frac{3}{2}.$$

NC5. Suppose f and g are continuous real valued functions on $[a, b]$ and are differentiable on (a, b). Assume that $g'(x) \neq 0$ for any $x \in (a, b)$. Prove that there exists $\xi \in (a, b)$ such that

$$\frac{f'(\xi)}{g'(\xi)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

NC6. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(0,0) = 0, \quad f(x,y) = \frac{xy}{x^2 + y^2}, \quad (x,y) \neq (0,0).$$

Prove that the directional derivative of f at $(0,0)$ exists in all directions. Is f continuous at $(0,0)$? Justify your answer.

NC7. Solve the differential equation

$$x^2(x^2 - 1)\frac{dy}{dx} + x(x^2 + 1)y = x^2 - 1.$$

NC8. Let f be a real valued function on $\mathbb{R}$. If for all real x,

$$f(x) + 3f(1-x) = 5$$

holds, then show that f is a constant function.

NC9. Let $f : [0,1] \to [0,\infty)$ be a continuous function. Let

$$a = inf_{0 \leq x \leq 1} f(x) \quad \text{and} \quad b = sup_{0 \leq x \leq 1} f(x).$$

For every positive integer m, define

$$c_m = [\int_0^1 (f(x))^m dx]^{1/m}.$$

Prove that $c_m \in [a,b]$, for all $m \geq 1$, $lim_{m\to\infty} c_m$ exists and find its value.

NC10. Let $f_1 : [0,4] \to [0,4]$ be defined by $f_1(x) = 3 - (x/2)$. Define $f_n(x) = f_1(f_{n-1}(x))$ for $n \geq 2$.

- Prove that $lim_{n\to\infty} f_n(0)$ exists.
- Find the set of all x such that $lim_{n\to\infty} f_n(x)$ exists and also find the corresponding limits.

NC11. Let m be a fixed integer greater than 2. Prove that all simple graphs having n ($n \geq 3$) vertices and with m edges are connected if and only if $m > \binom{n-1}{2}$.

NC12. Suppose the collection $\{A_1, \cdots, A_k\}$ forms a group under matrix multiplication, where each A_i is an $n \times n$ real matrix. Let $A = \sum_{i=1}^k A_i$.

- Show that $A^2 = kA$.
- If the trace of A is zero, then show that A is the zero matrix.

NC13. Let A be an $n \times n$ integer matrix whose entries are all even. Show that the determinant of A is divisible by 2^n. Hence or otherwise, show that if B is an $n \times n$ matrix whose entries are ± 1, then the determinant of B is divisible by 2^{n-1}.

NC14. Let

$$A = \begin{pmatrix} 1 & 2 & 1 & -1 \\ 2 & 0 & t & 0 \\ 0 & -4 & 5 & 2 \end{pmatrix}.$$

If rank(A) $= 2$, calculate t.

NC15. Let $n,\ r,\ s$ be positive integers, each greater than 2. Prove that $n^r - 1$ divides $n^s - 1$ if and only if r divides s.

NC16. Let $\Omega = \{1, 2, 3, \ldots, 100\}$. In how many ways

$$a_1 < a_2 < a_3 < a_4 < a_5, \quad a_i \in \Omega$$

can be chosen from Ω such that $a_{i+1} - a_i \geq 2$ for each i ?

NC17. Show that $5 \mid x \mid + x(x-2) \geq 0$ for every real number x.

NC18. Let $N = 1! + 2! + \cdots + 2020!$. Find the remainder obtained when N is divided by 8.

NC19. Let G be a finite group and H the only subgroup of G of order $|H|$. Prove that H is normal in G.

NC20 Let H and K be subgroups of a group G of finite indices (i.e., $[G : H] < \infty$ and $[G : K] < \infty$). Prove that $H \cap K$ is also of finite index (i.e., $[G : H \cap K] < \infty$).

NC21. Consider all the permutations of the numbers $1, 2, \ldots, 9$. Find the number of permutations which satisfy all of the following:

- the sum of the numbers lying between 1 and 2 (including 1 and 2) is 12,
- the sum of the numbers lying between 2 and 3 (including 2 and 3) is 23,
- the sum of the numbers lying between 3 and 4 (including 3 and 4) is 34,
- the sum of the numbers lying between 4 and 5 (including 4 and 5) is 45.

NC22. If $\alpha,\ \beta,\ \gamma$ are the roots of the equation $x^3 + 6x + 1 = 0$, then prove that

$$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} + \frac{\beta}{\gamma} + \frac{\gamma}{\beta} + \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} = -3.$$

NC23. Let $X \sim Bin\ (n, p)$, and $Y \sim Poisson\ (\lambda)$. Let

$$T = X_1 + X_2 + \cdots + X_Y,$$

with X_i's i. i. d. $Bin\ (n, p)$ (and independent to Y), and

$$S = Y_1 + Y_2 + \cdots + Y_X,$$

with Y_i's i. i. d. $Poisson\ (\lambda)$ (and independent to X). Compare Expectations of T and S and Variances of T and S.