INDIAN STATISTICAL INSTITUTE

Periodical Examination

M. Tech (CS) - I Year (Semester - I)

Discrete Mathematics

Date : 12.09.2008 Maximum Marks : 60 Duration : 3 Hours

Note : You may answer any part of any question, but maximum you can score is 60. Be precise in your answers. This is a two page question paper.

Q 1 $f(n) \prec g(n)$ denotes $f(n) = o(g(n))$. Using this notation, find the hierarchy of the following functions: $\log^2 n$, 2^{n^2}, $\log\log n$, $n!$, 2^n, $n^{4/5}$, $\sqrt{n}$; and fill up the following in your answer sheet. 7

	$\prec$		$\prec$		$\prec$		$\prec$		$\prec$		$\prec$	

Ans 1: The hierarchy is as follows.

$\log\log n$	$\prec$	$\log^2 n$	$\prec$	$\sqrt(n)$	$\prec$	$n^{4/5}$	$\prec$	2^n	$\prec$	$n!$	$\prec$	2^{n^2}

Q 2: Prove or disprove the statement that $a^n = \Theta(b^n)$, where $b > a$ 4

Ans 2: The statement is wrong. For a function $f(n)$ to be $\Theta(g(n))$,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c,$$

where c is a constant strictly greater than 0. Now, consider the limit

$$\lim_{n\to\infty} \frac{a^n}{b^n}.$$

As $\frac{a}{b} < 1$, the above limit becomes 0.

Q 3: Draw the Hasse diagram of the set of all subsets of $\{1, 2, 3\}$ ordered by inclusion. As an example, $\emptyset$ (null set) is included in $\{2\}$. In the diagram, draw the relation of *immediate predecessor* by lines/arrows. 4

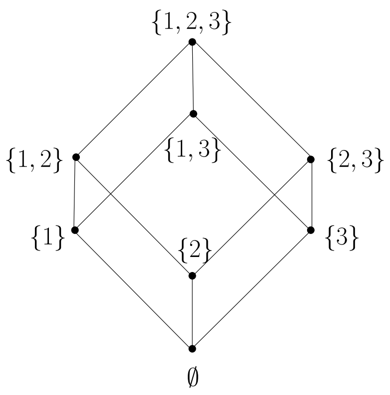


Figure 1: Hasse diagram of the set of all subsets of $\{1, 2, 3\}$ ordered by inclusion.

Ans 3: The Hasse diagram is as shown in Figure 1.

Q 4 Solve the recurrence $f(n) = 2f(\lfloor\sqrt{n}\rfloor) + \log_2 n$. You can make valid assumptions on n. If you need, you can also assume suitable initial conditions. 7

Ans 4: Let $m = \log n$. Therefore, the given recurrence becomes $T(2^m) = 2T(2^{m/2}) + m$. Now, let $S(m) = T(2^m)$. So, the new recurrence becomes $S(m) = 2S(m/2) + m$. Solving, we get $S(m) = O(m \log m)$. Therefore, $T(n) = T(2^m) = S(m) = O(m \log m) = O(\log n \log\log n)$.

Q 5 Let $\mathcal{R}$ be a relation from A to B where A and B are finite non-empty sets. Show that there exists $a \in A$ with $|\mathcal{R}(a)| \geq |\mathcal{R}|/|A|$. 4

Ans 5: Assume for a contradiction $|\mathcal{R}(a)| < |\mathcal{R}|/|A|\ \forall a \in A$. Then,

$$\begin{aligned} |R| &= \sum_{a \in A} |R(a)| \\ &< \sum_{a \in A} \frac{|R|}{|A|} \\ &= |A| \cdot \frac{|R|}{|A|} \\ &= |R| \end{aligned}$$

Thus, we arrive at a contradiction $|R| < |R|$. Hence, there exists $a \in A$ with $|\mathcal{R}(a)| \geq |\mathcal{R}|/|A|$.

Q 6 Find the negation of the proposition $p : 0 \leq x < \pi$. 2

Ans 6: The negation of p is the proposition $\sim p : x < 0$ or $x \geq \pi$.

Q 7 Determine if the proposition $(p \wedge q) \vee (\sim p \vee \sim q)$ is a tautology. 4

Ans 7: We determine whether the proposition is a tautology using the following truth table.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(p \wedge q) \vee (\sim p \vee \sim q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

Q 8 Show that $\sim p \rightarrow \sim q \equiv q \rightarrow p$ without using truth table. 4

Ans 8: The key idea is to use De Morgan's law.

$$\begin{aligned} \sim p \rightarrow \sim q &\equiv p \vee \sim q \\ &\equiv \sim (\sim p \wedge q) \\ &\equiv \sim (q \wedge \sim p) \\ &\equiv \sim q \vee \sim\sim p \\ &\equiv \sim q \vee p \\ &\equiv q \rightarrow p. \end{aligned}$$

Q 9 Use a recursion tree to give an asymptotically tight solution (in terms of Θ) to the recurrence $f(n) = f(\alpha n) + f((1-\alpha)n) + cn$ where α is a constant in the range $0 < \alpha < 1$ and $c > 0$ is also a constant. 6+6=12

[Hints: To show that $f(n) = \Theta(g(n))$, you have to show $f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$]

Ans 9: Assume without loss of generality that $\alpha \geq 1 - \alpha$ (otherwise we could just swap the terms $f(\alpha n)$ and $f((1-\alpha)n)$), so we have $1 > \alpha \geq 1/2$. Now, refer Figure 2 for the recursion tree. There are $\log_{\left(\frac{1}{1-\alpha}\right)} n$ full levels each taking cn. After $\log_{\left(\frac{1}{\alpha}\right)} n$ levels, the problem size is down to 1; where each level takes less than equal to cn. So, our guess for the lower bound is $\Omega(n \log_{\left(\frac{1}{1-\alpha}\right)} n) = \Omega(n \log n)$ and our guess for the upper bound is $O(n \log_{\left(\frac{1}{\alpha}\right)} n)$ $= O(n \log n)$. To prove the lower bound, we need to show that $f(n) \geq dn \log n$ where $d > 0$

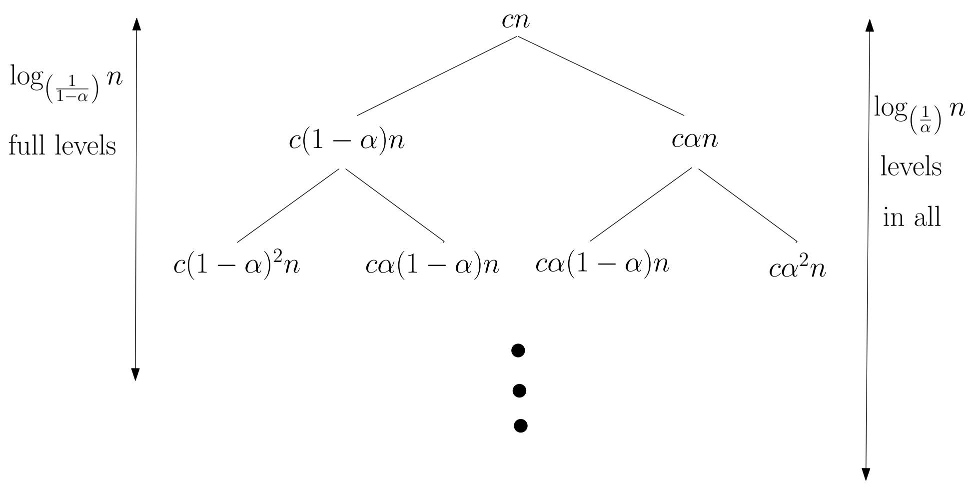


Figure 2: The recursion tree.

is a constant.

$$\begin{aligned} f(n) &= f(\alpha n) + f((1-\alpha)n) + cn \\ &\geq d\alpha n \log(\alpha n) + d(1-\alpha)n \log((1-\alpha)n) + cn \\ &= d\alpha n \log \alpha + d\alpha n \log n + d(1-\alpha)n \log(1-\alpha) + d(1-\alpha)n \log n + cn \\ &= dn \log n + dn(\alpha \log \alpha + (1-alpha)\log(1-alpha)) + cn \\ &\geq dn \log n. \end{aligned}$$

if $dn(\alpha \log \alpha + (1-\alpha)\log(1-\alpha)) + cn \geq 0$, i.e. $d(\alpha \log \alpha + (1-\alpha)\log(1-\alpha)) + c \geq 0$. From this inequality, the constant d for the lower bound can be determined.

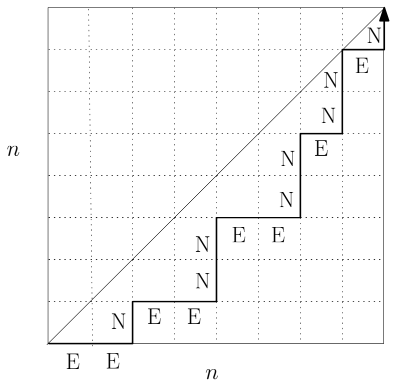
Similarly, for the upper bound we have

$$\begin{aligned} f(n) &= f(\alpha n) + f((1-\alpha)n) + cn \\ &\leq d\alpha n \log(\alpha n) + d(1-\alpha)n \log((1-\alpha)n) + cn \\ &= d\alpha n \log \alpha + d\alpha n \log n + d(1-\alpha)n \log(1-\alpha) + d(1-\alpha)n \log n + cn \\ &= dn \log n + dn(\alpha \log \alpha + (1-alpha)\log(1-alpha)) + cn \\ &\leq dn \log n. \end{aligned}$$

if $dn(\alpha \log \alpha + (1-\alpha)\log(1-\alpha)) + cn \leq 0$, i.e. $d(\alpha \log \alpha + (1-\alpha)\log(1-\alpha)) + c \leq 0$. From this inequality, the constant d for the upper bound can be determined.

Thus, $f(n) = O(n \log n)$ and $f(n) = \Omega(n \log n)$. So, we have $f(n) = \Theta(n \log n)$.

Q 10 You have an $n \times n$ square grid $\mathcal{A}$. A monotonic path, as shown in Figure 3, from the lower left corner to the upper right corner is a path along the horizontal and vertical edges of the grid that do not cross the diagonal, i.e., it stays below the diagonal. Deduce out a recurrence



E E N E E N N E E N N E N N E N

Figure 3: A monotonic path in an $n \times n$ square grid.

formula for counting the number of such monotonic paths. Only the recurrence equation would be enough. No need of solving the recurrence. 7

Ans 10: Let $f(n)$ be the number of monotonic paths in an $n \times n$ square grid. See Figure 3. Any monotonic path can be encoded as a string of length $2n$ using alphabets E (meaning movement towards East) and N (meaning movement towards North). The property of any such monotonic path is that considering from the start point (at the left bottom corner) the number of east bound movements will be greater than or equal to the number of north bound movements. So, in terms of the string it is as follows. No initial part of the string has more Ns than Es. Thus, any such string of length greater than or equal to 2 can be written as $\mathsf{E}s_1\mathsf{N}s_2$, where s_1 and s_2 are string representations of valid monotonic paths. s_1 and s_2 may be null strings also. Let the lengths of s_1 and s_2 be $2n_1$ and $2n_2$ respectively. So, $2n_1 + 2n_2 + 2 = 2n$; i.e. $n_1 + n_2 + 1 = n$. This surely implies that n_1 and n_2 can vary from 0 to $n-1$. So, we can write the recurrence for $f(n)$ as

$$f(n) = f(0) * f(n-1) + f(1) * f(n-2) + \ldots + f(n-1) * f(0)$$

$$= \sum_{i=0}^{n-1} f(i) * f(n-i-1)$$

where $f(i)$ is the number of strings for s_1 and $f(n-i-1)$ is the number of strings for s_2. Thus, together with the initial condition, the recurrence is

$$f(n) = \begin{cases} \sum_{i=0}^{n-1} f(i) * f(n-i-1) & \text{if } n \geq 2; \\ 1 & \text{if } n = 1. \end{cases}$$

If you are a bit hassled with the term $f(n-i-1)$, get rid of it as follows. Put $n = m+1$. So,

$$\begin{aligned} f(n) &= \sum_{i=0}^{n-1} f(i) * f(n-i-1) \\ f(m+1) &= \sum_{i=0}^{m} f(i) * f(m-i). \end{aligned}$$

This recurrence solves to the Catalan number.

Q 11 Prove or disprove the following statement. In any graph $G = (V, E)$ with a finite number of vertices, there always exist two vertices with the same degree. 5

Ans 11 Let $|V| = n$. So, each vertex in G can have as degree any one of the integers in $[0, n-1]$. But, if a vertex has a degree 0, then another vertex cannot have a degree of $n-1$ and vice versa. That is, 0 and $n-1$ both are not possible as degrees. Therefore, the list of degrees of n vertices can have at most $n-1$ distinct numbers. Now, apply pigeonhole principle with n vertices as pigeons and $n-1$ numbers as holes to show that there always exist two vertices with the same degree.

Q 12 Find the number of solutions to the following equation using generating functions:

$$x_1 + x_2 + x_3 = 14$$

where $x_1, x_2, x_3 \geq 0$, x_1 is even, $2 \leq x_2 \leq 7$, and x_3 is prime. 5+2=7

[Note: Out of 7, 5 is for the general form of the solution using generating function. 2 is for the exact expression.]

Ans 12: The number of solutions is the co-efficient of x^{14} in the polynomial associated with a generating function. The contribution of x_1 (being even) is the polynomial $(x^2 + x^4 + x^6 +$

$x^8 + x^{10} + x^{12})$; the contribution of x_2 ($2 \leq x_2 \leq 7$) is $(x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$; and the contribution of x_3 (x_3 being prime) is $(x^2 + x^3 + x^5 + x^7 + x^{11} + x^{13}))$. So, the final polynomial for the generating function in which the coefficient of x^{14} is to be searched is:

$$(x^2+x^4+x^6+x^8+x^{10}+x^{12})(x^2+x^3+x^4+x^5+x^6+x^7)(x^2+x^3+x^5+x^7+x^{11}+x^{13})$$

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