Shortest Path: Dijkstra's algorithm

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1 Introduction

Induction plays a major role in algorithm design and analysis. In algorithms based on induction, the process of constructing proofs of the induction and the algorithm design are very similar.

Mathematical induction basically works as follows. Suppose, we want to prove a statement $\mathcal{P}$ with a parameter $n \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. Instead of proving $\mathcal{P}$ directly, we show that $\mathcal{P}$ holds for:

(i) any base case (say $n = 1$)

(ii) $\forall n > 1$, if $\mathcal{P}$ holds for $n - 1$, then $\mathcal{P}$ holds for n.

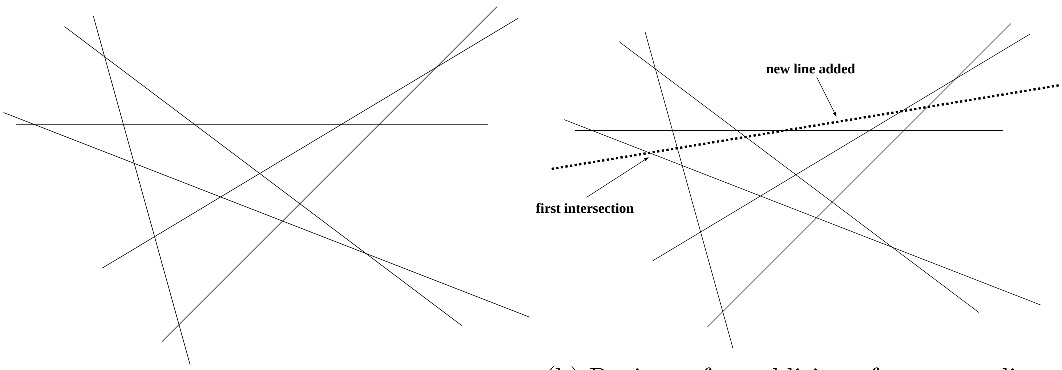
For some more variants of this induction technique, see [1].

2 Some examples

2.1 Counting regions in the plane

A set $\mathcal{L} = \{\ell_\infty, \ell_\in, \dots, \ell_\backslash\}$ of n lines in the plane are said to be in the **general position** if (i) no two lines are parallel and (ii) no three lines intersect at a common point. The problem is to find the number of regions $\mathcal{R}^n$ formed by n lines in general position. What better technique to use than induction? We will discuss here a proof that is different from the one discussed in class.

The idea is as follows. Let the number of regions formed after adding the $(n-1)^{th}$ line be $\mathcal{R}^{n-1}$. We try to find out the number of new regions r_n formed after adding the n^{th} line. Then, surely $\mathcal{R}^n = \mathcal{R}^{n-1} + r_n$. We now try to find out r_n.



(a) Regions in the presence of $n - 1$ lines.

(b) Regions after addition of one more line.

Figure 1: Region formation for lines in general position.

Lemma 1 *Adding the n^{th} line to the already existing $(n-1)$ lines, the number of new regions added is n.*

Proof: The n^{th} line can cut each of the previously existing $(n-1)$ lines once. The line after each such intersection breaks an existing region into two and gives rise to a new region after the intersection; i.e. we have $(n-1)$ regions. Add to that one more region formed before the first intersection. Thus, we have n new regions. See Fig. 1 for an illustration. □

Theorem 1 *The number of regions in the plane formed by n lines in general position is $\frac{n(n+1)}{2}+1$.*

Proof: We proceed to prove this by induction. Let us first look into the base case.

Base case: The first line introduces two new regions.

Induction Hypothesis: For every $m > 1$, the number of new regions added is m. **Proof:** Assuming that m lines have added m new regions, the $(m+1)^{th}$ line by Lemma 1 adds $(m+1)$ new regions.

So, the total number of regions $\mathcal{R}^n = 2 + \sum_{i=2}^{n} = \frac{n(n+1)}{2} + 1$. □

2.2 A colouring problem

See [1] for this problem.

2.3 Euler's formula for a planar connected graph

Let $\mathcal{G} = \{V, E\}$ be a graph where V denotes the set of vertices with $|V| = v$ and E denotes the set of edges with $|E| = e$. $\mathcal{G}$ is planar if it can be embedded in the plane without any edge crossing. Now, let us try to look into the characterization of non-planar garphs. Non-planar graphs are those for which planar embeddings imply edge crossings. Loosely speaking, if any graph $\mathcal{G}$ has anyone of a K_5 or $K_{3,3}$ as its subgraph, then $\mathcal{G}$ is non-planar. A K_5 is a complete graph of 5 vertices and a $K_{3,3}$ is a complete bipartite graph of 3 vertices on each side of the partition.

A planar graph has an important relation between its edges, vertices and faces. Let the number of faces or regions in the planar embedding of $\mathcal{G}$ be f. Then, Euler's formula states that $v - e + f = 2$.

Theorem 2 *Let $\mathcal{G}$ be a planar connected graph having v vertices, e edges and f faces. Then, Euler's formula states that $v - e + f = 2$.*

Proof: We proceed to prove this by induction. But, as the problem has more than two parameters, we need to judiciously select the parameter on which we will use induction. We do the induction on the number of edges.

Base case: If $e = 1$, then we have two vertices and one face. So, it is trivial.

Induction hypothesis: Euler's formula holds for planar connected graphs with $e \leq n-1$, i.e. $v - (n-1) + f = 2$.

Now, we prove that Euler's formula holds for $e = n$. i.e. when we add one more edge. The new edge can be added in two ways. In the first case, an edge joins an existing vertex with a new one. So, no new faces are created. The formula becomes $(v+1) - n + f$, which can be rewritten as $v - (n-1) + f$ which is equal to 2 by the induction hypothesis. In the second case, an edge joins two existing vertices, thereby creating one new face. The formula becomes $v - n + (f+1)$, which can be rewritten as $v - (n-1) + f$ which is again equal to 2 by the induction hypothesis. $\square$

References

[1] J. Kleinberg and É. Tardos, *Algorithm Design*, Pearson Education, 2006.

[2] T. H. Cormen, C. E. Leiserson, R. L. Rivest, C. Stein , *Algorithms - Design Techniques and Analysis*, Prentice Hall of India.