INDIAN STATISTICAL INSTITUTE

Class Test I

M. Tech (CS) - I Year (Semester - I)

Discrete Mathematics

Date : 9.9.2009 Maximum Marks : 25 Duration : 1.5 Hours

This is a closed notes exam.

Note : You may answer any part of any question, but maximum you can score is 25.

(Q1) For $n = 2^{2^k}$, $k \geq 1$, do proper substitution to bring the following recurrence to a form you know and has been discussed in the class.

$$T(n) = \begin{cases} d & \text{if } n = 2; \\ 2T(\sqrt{n}) + b\log n & \text{if } n > 2. \end{cases}$$

[4+2=6]

[Note: Out of 6, 4 is for proper substitution and 2 is for solving it.]

(Ans:) With $n = 2^{2^k}$, $2^k = \log n$ and the recurrence after substitution is as follows:

$$T(2^{2^k}) = \begin{cases} d & \text{if } k = 0; \\ 2T(2^{2^{k-1}}) + b2^k & \text{if } k > 0. \end{cases}$$

Let $T_1(k) = T(2^{2^k})$. So, now we have the recurrence as

$$T_1(k) = \begin{cases} d & \text{if } k = 0; \\ 2T_1(k-1) + b2^k & \text{if } k > 0. \end{cases}$$

This recurrence is of the form $T(k) = G(k)T(k-1) + H(k)$ for $k > 1$ with the initial condition $T(0) = d$. As was discussed in the class, this solves to

$$T(k) = (\Pi_{i=1}^{k} G(i)) \left(T(0) + \sum_{i=1}^{k} \frac{H(i)}{\Pi_{j=1}^{i} G(j)} \right).$$

Here, $G(k) = 2$ and $H(k) = b\,2^k$. So $T(k) = 2^k \left(b + \frac{b2^2}{2^2} + \ldots + \frac{b2^k}{2^k} \right) = 2^k(d + bk)$. Now, replace back $2^k = \log n$, to get $T(n) = d\log n + b\log n \log\log n$.

(Q2) Determine if the proposition $(p \wedge q) \vee (\sim p \vee \sim q)$ is a tautology. [3]

(Ans:) We determine whether the proposition is a tautology using the following truth table.

p	q	$p \wedge q$	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(p \wedge q) \vee (\sim p \vee \sim q)$
T	T	T	F	F	F	T
T	F	F	F	T	T	T
F	T	F	T	F	T	T
F	F	F	T	T	T	T

(Q3) Show that $\sim p \rightarrow \sim q \equiv q \rightarrow p$ without using truth table. [4]

(Ans:) The key idea is to use De Morgan's law.

$$\begin{aligned} \sim p \rightarrow \sim q &\equiv p \vee \sim q \\ &\equiv \sim (\sim p \wedge q) \\ &\equiv \sim (q \wedge \sim p) \\ &\equiv \sim q \vee \sim\sim p \\ &\equiv \sim q \vee p \\ &\equiv q \rightarrow p. \end{aligned}$$

(Q4) $f(n) \prec g(n)$ denotes $f(n) = o(g(n))$. Using this notation, find the hierarchy of the following functions: $\log^2 n$, 2^{n^2}, $\log \log n$, $n!$, 2^n, $n^{4/5}$, $\sqrt{n}$; and fill up the following in your answer sheet. [7]

	$\prec$		$\prec$		$\prec$		$\prec$		$\prec$		$\prec$	

(Ans:) The hierarchy is as follows.

$\log \log n$	$\prec$	$\log^2 n$	$\prec$	$\sqrt(n)$	$\prec$	$n^{4/5}$	$\prec$	2^n	$\prec$	$n!$	$\prec$	2^{n^2}

(Q5) How many 8-digit sequences consisting of digits 0 through 9 are there? How many of them contain an even number of odd digits? Prove your results. [2+2=4]

(Ans:) The number of 8-digit sequences is 10^8. This is easy to see as the number of 8-digit sequences has a bijection with $\{0, 1, \ldots, 10^8 - 1\}$.

Let A_e be the set of all sequences containing an even number of odd digits and A_o be the set of all sequences containing an odd number of odd digits. Notice that, $|A_e| + |A_o| = 10^8$. If we can set up a bijection between A_e and A_o, then we are done. To set up a bijection f, take any sequence $s \in A_e$ and change the first digit to its next number modulo 9, i.e. 0 to 1, 1 to 2, . . ., 9 to 0. The modified sequence $f(s)$ has an odd number of odd digits. So, f is a mapping from A_e to A_o. To see that f is one-to-one, notice that two different sequences $s_1, s_2 \in A_e$ cannot map to the same sequence in A_o. To see that f is onto, take any $s' \in A_o$ and see that it can be obtained as $f(s)$ for some $s \in A_e$ by changing 1 to 0, 2 to 1, . . ., 9 to 8. So, f is a bijective map. Hence, $|A_e| = |A_o| = 5 \cdot 10^7$.

Another way of counting is as follows. There are five odd digits - $\{1, 3, 5, 7, 9\}$. We are asked to find out the 8-digit sequences containing an even number of odd digits, i.e., there can be 0, 2, 4, 6 or 8 occurrences of odd digits. Any i ($i \in \{0, 2, 4, 6, 8\}$) occurrences of odd digits can occur in $\binom{8}{i}$ ways and these i places can be filled with the five odd digits - $\{1, 3, 5, 7, 9\}$ in 5^i ways. The rest $8-i$ places can be filled with five even digits - $\{0, 2, 4, 6, 8\}$ in 5^{8-i} ways. Thus, the total number of 8-digit sequences with i odd digits is $\binom{8}{i}5^8$. So, the total number of 8-digit sequences containing an even number of odd digits is $\binom{8}{0}5^8 + \binom{8}{2}5^8 + \binom{8}{4}5^8 + \binom{8}{6}5^8 + \binom{8}{8}5^8$ $= 128 \cdot 5^8 = 5 \cdot 10^7$.

(Q6) Find the number of solutions to the following equation using generating functions:

$$x_1 + x_2 + x_3 = 14$$

where $x_1, x_2, x_3 \geq 0$, x_1 is even, $2 \leq x_2 \leq 7$, and x_3 is prime. [4+2=6]

[Note: Out of 6, 4 is for the general form of the solution using generating function. 2 is for the exact expression.]

(Ans:) The number of solutions is the co-efficient of x^{14} in the polynomial associated with a generating function. The contribution of x_1 (being even) is the polynomial $(x^2 + x^4 + x^6 + x^8 + x^{10} + x^{12})$; the contribution of x_2 ($2 \leq x_2 \leq 7$) is $(x^2 + x^3 + x^4 + x^5 + x^6 + x^7)$; and the contribution of x_3 (x_3 being prime) is $(x^2 + x^3 + x^5 + x^7 + x^{11} + x^{13}))$. So, the final polynomial for the generating function in which the coefficient of x^{14} is to be searched is:

$$(x^2+x^4+x^6+x^8+x^{10}+x^{12})(x^2+x^3+x^4+x^5+x^6+x^7)(x^2+x^3+x^5+x^7+x^{11}+x^{13})$$

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