INDIAN STATISTICAL INSTITUTE

Class Test

M. Tech (CS) - I Year (Semester - I), 2009-2010

Discrete Mathematics

Date : 03.12.2009   Maximum Marks : 20   Duration : 1.5 Hours

This is a closed notes exam.

Note : You may answer any part of any question, but maximum you can score is 20.

(Q1) An edge and a vertex in a connected graph $G = (V, E)$ are said to *cover* each other if they are incident. Two edges are said to be adjacent if they share a vertex. A set of edges in G that covers all the vertices in G is said to be an *edge cover* of G. Let α_1 denote the minimum number of edges in any *edge cover* of G. An *independent set of edges* in G has no two of its edges adjacent. Let β_1 denote the maximum number of edges in any *independent set of edges* of G. Now, prove or disprove the following statement: $\alpha_1 + \beta_1 = |V|$. [5]

(Ans 1:) We will need the following definition.

Definition 1 *A star is a complete bipartite graph $K_{1,n}$.*

Consider an independent set E_{mis} of β_1 edges. An edge cover E_{ec} can be produced by taking the union of E_{mis} and a set of edges where each edge covers a vertex in G that was not covered by any edge belonging to E_{mis}. Note that $|E_{mis}| + |E_{ec}| \leq |V|$. Also, $E_{ec} \geq \alpha_1$. These two inequalities taken together give us the inequality $\alpha_1 + \beta_1 \leq |V|$.

Now, consider a minimum edge cover E_{mec} of G. E_{mec} cannot contain an edge both of whose end vertices are incident with edges also in E_{mec}. This implies that E_{mec} is the sum of *stars* of G considered as sets of edges. Now, choose one edge from each of these stars to obtain an independent set E_{is} of edges. Clearly, $E_{is} \leq \beta_1$. Also, $|E_{mec}| + |E_{is}| = |V|$. These two inequalities taken together give us $\alpha_1 + \beta_1 \geq |V|$.

So, finally we have $\alpha_1 + \beta_1 = |V|$.

(Q2) For any well formed formulas A and B, show that $(\sim A \Rightarrow (A \Rightarrow B))$ is a theorem of the axiom system for propositional calculus. [5]

(Ans 2:) We need to show $\vdash_L (\sim A \Rightarrow (A \Rightarrow B))$.

(1)	$\sim A$	hypothesis
(2)	A	hypothesis
(3)	$A \Rightarrow (\sim B \Rightarrow A)$	Axiom 1
(4)	$\sim A \Rightarrow (\sim B \Rightarrow \sim A)$	Axiom 1
(5)	$\sim B \Rightarrow A$	2, 3 and MP
(6)	$\sim B \Rightarrow \sim A$	1, 4 and MP
(7)	$(\sim B \Rightarrow \sim A) \Rightarrow ((\sim B \Rightarrow A) \Rightarrow B)$	Axiom 3
(8)	$(\sim B \Rightarrow A) \Rightarrow B$	6, 7 and MP
(9)	B	5, 8 and MP

Thus, $\{\sim A, A\} \vdash_L B$. Apply deduction theorem twice to get $(\sim A \Rightarrow (A \Rightarrow B))$.

(Q3) Let $G = (V, E)$ with $|V| = n$ be a connected graph. Let the maximum independent set of G be $\beta(G)$ and the chromatic number of G be $\chi(G)$. Prove that $n \leq \beta(G)\chi(G)$. Use this result to show that $\beta(G) \geq n/4$ for a planar graph. [5]

[Hints: Sometimes question in Graph Theory has its solution elsewhere!]

(Ans 3:) Partition the vertices of G by their colors. So, n vertices are distributed among $\chi(G)$ colors. Now, apply pigeonhole principle with n vertices as pigeons and $\chi(G)$ colors as holes. So, one of the colors must contain $n/\chi(G)$ vertices. By the definition of coloring in a graph, these $n/\chi(G)$ vertices should be pairwise nonadjacent, i.e. they should not have edges. So, these $n/\chi(G)$ vertices form an independent set. Therefore, the maximum independent set $\beta(G) \geq n/\chi(G)$. So, $n \leq \beta(G)\chi(G)$.

The famous four color theorem for a planar graph states that a planar graph can be colored using at most four colors, i.e. $\chi(G) \leq 4$. Using the relation $\beta(G) \geq n/\chi(G)$ and $\chi(G) \leq 4$, we have $\beta(G) \geq n/4$ for a planar graph.

(Q4) Let $G = (V, E)$ with $|V| = n$ be a spanning tree on n nodes. Reason out why G is not Eulerian. Suggest a scheme by which you can alter G to make it Eulerian. The only operation of alteration allowed is addition of edges to E. The altered graph should be connected. Your marks obviously will depend on the number of edges added. [1+2]

[Hints: Making the altered graph a complete graph on n vertices is a very inefficient solution! Multigraphs are allowed.]

(Ans 4:) G ia a spanning tree and cannot be Eulerian because it surely has a vertex with degree equal to one, i.e., G has an odd degree vertex.

To transform G into an Eulerian graph, just double each edge. That is, for each edge $e(= (u, v)) \in E$, add another edge (u, v). Now, each vertex has even degree. So, the altered graph is Eulerian. This is a very simple and elegant scheme, though there can be other methods.

Another method is as follows. Consider the odd degree vertices in the spanning tree. These are the vertices that do not allow the graph to be Eulerian. But we know that the number of odd degree vertices is even. Let the set of odd degree vertices be $X \subseteq V$. Now, consider the complete graph $K_{|X|}$ on the vertex set X. Perform a matching on $K_{|X|}$. Let the set of edges in the matching be M. Add the edges in M to the spanning tree. This will make the degree of each vertex that had odd degree to be even.

(Q5) Let $G = (V, E)$ be a connected graph. Let P be the longest path in G between two vertices $v_1, v_2 \in V$. P can be decomposed as follows: $v_1 \rightsquigarrow x \rightsquigarrow y \rightsquigarrow v_2$, where $\rightsquigarrow$ denotes a path between two vertices. Now, consider the following statement: The sub-path of P between x and y is also the longest path between x and y. If the above statement is true, prove it; else, disprove it. [2]

(Ans 5:) The statement is wrong. A counterexample is as follows. Let the path P with the corresponding path lengths (shown above $\rightsquigarrow$) be

$$v_1 \overset{l_1}{\rightsquigarrow} x \overset{l_2}{\rightsquigarrow} y \overset{l_3}{\rightsquigarrow} v_2.$$

Let there be another path from v_1 to v_2 that avoids x and y with path length l, i.e. $v_1 \overset{l}{\rightsquigarrow} v_2$. Now, for a graph it can so happen that $l_1 + l + l_3 > l_2$, making the longest path from x to y not to be a sub-path of P.

(Q6) Show that $\{A \Rightarrow (B \Rightarrow C), B\} \vdash (A \Rightarrow C)$ where $\vdash$ has the usual meaning. [5]

(Ans 6:)

(1)	$A \Rightarrow (B \Rightarrow C)$	hypothesis
(2)	A	hypothesis
(3)	B	hypothesis

(4)	$B \Rightarrow C$	1, 2 and MP
(5)	C	3, 4 and MP

Thus, $\{A \Rightarrow (B \Rightarrow C),\ B,\ A\} \vdash_L C$. Apply deduction theorem once to get $\{A \Rightarrow (B \Rightarrow C), B\} \vdash (A \Rightarrow C)$.