# INDIAN STATISTICAL INSTITUTE

**Periodical Examination**

M. Tech (CS) - I Year, 2009-2010 (Semester - I)

*Discrete Mathematics*

Date : 16.09.2009 Maximum Marks : 50 Duration : 3 Hours

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Note: This is a **cheat sheet based** examination. You can carry with yourself **two A4 sized sheets** with your name and roll number written neatly on top of both the sheets. You have to submit the cheat sheets after the examination is over. Cheat sheets cannot be shared.

You may answer any part of any question, but maximum you can score is 50.

This is a two page question paper.

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**(Q 1)** Prove or disprove the statement that $a^n = \Theta(b^n)$, where $b > a$ [4]

**(Ans 1:)** The statement is wrong. For a function $f(n)$ to be $\Theta(g(n))$,

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = c,$$

where $c$ is a constant strictly greater than 0. Now, consider the limit

$$\lim_{n\to\infty} \frac{a^n}{b^n}.$$

As $\frac{a}{b} < 1$, the above limit becomes 0.

**(Q 2)** Consider the statement: $2^{f_1(n)} = O(2^{f_2(n)})$. If it is true, prove it; else, disprove it. [4]

**(Ans 2:)** The above statement is false. Assume $f_1(n) = 2n$ and $f_2(n) = n$. So, $f_1(n) = O(f_2(n))$. Now, let $2^{f_1(n)} = O(2^{f_2(n)})$, i.e. $2^{2n} = O(2^n)$. If this is true, then $\exists$ constants $c, n_0 > 0$ such that $0 \leq 2^{2n} \leq c2^n$ which implies $c \geq 2^n$. But, then $c$ cannot be a constant which leads to a contradiction.

**(Q 3)** Determine if the propositional statement $(A \Rightarrow B) \Leftrightarrow \sim (A \wedge (\sim B))$ is a tautology, a contradiction or neither. [4]

**(Ans 3:)** We use the following truth table to show that the propositional statement is a tautology.

| $A$ | $B$ | $A \Rightarrow B$ | $\sim B$ | $A \wedge (\sim B)$ | $\sim (A \wedge (\sim B))$ | $(A \Rightarrow B) \Leftrightarrow \sim (A \wedge (\sim B))$ |
|---|---|---|---|---|---|---|
| T | T | T | F | F | T | T |
| T | F | F | T | T | F | T |
| F | T | T | F | F | T | T |
| F | F | T | T | F | T | T |

**(Q 4)** Prove or disprove the following statement: $(A \Leftrightarrow B) \Leftrightarrow C$ and $A \Leftrightarrow (B \Leftrightarrow C)$ are logically equivalent. [4]

**(Ans 4:)** To show that $(A \Leftrightarrow B) \Leftrightarrow C$ and $A \Leftrightarrow (B \Leftrightarrow C)$ are logically equivalent, we have to show that $((A \Leftrightarrow B) \Leftrightarrow C) \Leftrightarrow (A \Leftrightarrow (B \Leftrightarrow C))$ is a tautology.

| $A$ | $B$ | $C$ | $A \Leftrightarrow B$ | $B \Leftrightarrow C$ | $(A \Leftrightarrow B) \Leftrightarrow C$ | $A \Leftrightarrow (B \Leftrightarrow C)$ | $((A \Leftrightarrow B) \Leftrightarrow C) \Leftrightarrow (A \Leftrightarrow (B \Leftrightarrow C))$ |
|---|---|---|---|---|---|---|---|
| T | T | T | T | T | T | T | T |
| T | T | F | T | F | F | F | T |
| T | F | T | F | F | F | F | T |
| T | F | F | F | T | T | T | T |
| F | T | T | F | T | F | F | T |
| F | T | F | F | F | T | T | T |
| F | F | T | T | F | T | T | T |
| F | F | F | T | T | F | F | T |

**(Q 5)** Let $\mathcal{R}$ be a relation from $A$ to $B$ where $A$ and $B$ are finite non-empty sets. Show that there exists $a \in A$ with $|\mathcal{R}(a)| \leq |\mathcal{R}|/|A|$. [4]

**(Ans 5:)** Assume for a contradiction $|\mathcal{R}(a)| > |\mathcal{R}|/|A| \; \forall a \in A$. Then,

$$\begin{aligned} |R| &= \sum_{a \in A} |R(a)| \\ &> \sum_{a \in A} \frac{|R|}{|A|} \\ &= |A| \cdot \frac{|R|}{|A|} \\ &= |R| \end{aligned}$$

Thus, we arrive at a contradiction $|R| > |R|$. Hence, there exists $a \in A$ with $|\mathcal{R}(a)| \leq |\mathcal{R}|/|A|$.

**(Q 6)** Let $A = \{x \mid x \in \mathbb{R} \text{ and } 0 \leq x \leq 1\}$ and $B = \{y \mid y \in \mathbb{R} \text{ and } 0 \leq y \leq 2\}$ be two sets. Prove or disprove the statement that $|A| = |B|$. $\mathbb{R}$ denotes the set of reals. [4]

**(Ans 6:)** Define a function $f : A \rightarrow B$ as $y = f(x) = 2x$, for $x \in A$ and $y \in B$. It is easy to check that this function is a bijection as it is both one-to-one and onto. Now, recall Cantor-Schröder-Bernstein theorem which states that two sets are equinumerous if and only if there exists a bijection between them. As there exists a bijection between $A$ and $B$, so $|A| = |B|$.

**(Q 7)** In number theory, the *fundamental theorem of arithmetic* or the *unique prime factorization theorem* states that every positive integer (except the number 1) can be represented in exactly one way apart from rearrangement as a product of one or more primes. That is, any natural number $n$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \ldots, p_r$ are distinct primes and each $\alpha_i$ is a natural number. As an example, 1176 can be written as $2^3 \cdot 3 \cdot 7^2$. The Euler function corresponding to a natural number $n$ and denoted as $\varphi(n)$ is defined as follows:

$$\varphi(n) = \left|\{m \in \{1, 2, \dots, n\} |\ \gcd(n, m) = 1\}\right|.$$

That is $\varphi(n)$ is the number of natural numbers $m \leq n$ that are relatively prime to $n$.

(i) What is the value of $\varphi(n)$ when $n$ is prime?

(ii) Prove that for $n = p_1^{\alpha_1} p_2^{\alpha_2}, \dots p_r^{\alpha_r}$, we have

$$\varphi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right)$$

[2+6=8]

**(Ans 7 (i):)** For a prime number $n$, every $m$ ($1 \leq m < n$) is relatively prime to $n$. So, $\varphi(n) = n - 1$.

**(Ans 7 (ii):)** For an idea of the general solution, first consider the case when $n = p^\alpha$, where $p$ is a prime and $\alpha \in \mathbb{N}$. The numbers that are not relatively prime to $p^\alpha$ are the multiples of $p$ that are less than or equal to $p^\alpha$, i.e. $p$, $2p$, …, $p^{\alpha-1}$, $p^{\alpha-1}p$. How many such numbers are there? Obviously, there are $\frac{p^\alpha}{p} = p^{\alpha-1}$ such numbers. So, $\varphi(p^\alpha) = p^\alpha - p^{\alpha-1} = p^\alpha(1 - \frac{1}{p})$.

Now, we know that any natural number $n$ can be written as $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_i^{\alpha_i} \cdots p_r^{\alpha_r}$ where $p_1, p_2, \dots, p_i, \dots, p_r$ are distinct primes and each $\alpha_i \in \mathbb{N}$. As in the previous case, we need to find all multiples of the primes $p_i$. To that end, let $A_i$ denote all the multiples of the prime $p_i$, i.e. $A_i = \{m \in \{1, 2, \dots, n\} |\ p_i \text{ divides } m\}$. Now, we can surely write $\varphi(n) = n - |\bigcup_{i=1}^r A_i|$. So, we need to find out $|\bigcup_{i=1}^r A_i|$. What about the intersections of the sets $A_i$? As we are dealing with primes, any $A_i \cap A_j = \frac{n}{p_i p_j}$, where $i \neq j$. Similarly, any $A_i \cap A_j \cap A_k = \frac{n}{p_i p_j p_k}$, where $i < j < k$. Extending this further, we have $A_1 \cap A_2 \cap \dots \cap A_r = \frac{n}{p_1 p_2 \cdots p_r}$. So, we have

$$\begin{aligned}
\varphi(n) &= n - \left(\sum_{i=1}^{r} \frac{n}{p_i} - \sum_{1 \leq i < j \leq r} \frac{n}{p_i p_j} + \sum_{1 \leq i < j < k \leq r} \frac{n}{p_i p_j p_k} - \dots + (-1)^{r-1} \frac{n}{p_1 p_2 \cdots p_r}\right) \\
&= n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)\cdots\left(1 - \frac{1}{p_r}\right)
\end{aligned}$$

**(Q 8)** Prove or disprove the following statement. In any graph $G = (V, E)$ with a finite number of vertices, there always exist two vertices with the same degree. [4]

**(Ans 8:)** Let $|V| = n$. So, each vertex in $G$ can have as degree any one of the integers in $[0, n-1]$. But, if a vertex has a degree 0, then another vertex cannot have a degree of $n - 1$ and vice versa. That is, 0 and $n - 1$ both are not possible as degrees. Therefore, the list of degrees of $n$ vertices can have at most $n - 1$ distinct numbers. Now, apply pigeonhole principle with $n$ vertices as pigeons and $n - 1$ numbers as holes to show that there always exist two vertices with the same degree.

**(Q 9)** Solve the following recurrence assuming $n = 2^k$, where $k$ is a positive integer.

$$T(n) = \begin{cases} T(n/2) + T(n/4) & \text{if } n > 4; \\ 1 & \text{if } n = 4; \\ 1 & \text{if } n = 2 \end{cases}$$

[5]

**(Ans 9:)** Putting $n = 2^k$ in the above recurrence, we observe

$$T(2^k) = \begin{cases} T(2^{k-1}) + T(2^{k-2}) & \text{if } 2^k > 4; \\ 1 & \text{if } 2^k = 4; \\ 1 & \text{if } 2^k = 2 \end{cases}$$

Substituting $T(2^k) = f(k)$, the recurrence becomes

$$f(k) = \begin{cases} f(k-1) + f(k-2) & \text{if } k > 2; \\ 1 & \text{if } k = 2; \\ 1 & \text{if } k = 1 \end{cases}$$

This is nothing but the *fibonacci sequence* that you know very well to solve.

$$f(k) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^k$$

Replacing $k = \log n$, we have

$$T(n) = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{\log n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{\log n}$$

**(Q 10)** Solve the following recurrence assuming $c$ to be a positive constant.

$$T(n) \le \begin{cases} T(\lfloor n/5 \rfloor) + T(\lfloor 3n/4 \rfloor) + cn & \text{if } n \ge 44; \\ c & \text{if } n < 44; \end{cases}$$

[5]

**(Ans 10:)** The crucial point to note about this recurrence is that we are throwing away a constant fraction of the data at each step of the recursion as $\frac{1}{5} + \frac{3}{4} < 1$ and as we know by now this recurrence solves to $\Theta(n)$.

For a guess, assume $T(n) \le bcn$ for a suitable positive constant $b$. Substituting, we obtain

$$\begin{aligned} T(n) &= T(\lfloor n/5 \rfloor) + T(\lfloor 3n/4 \rfloor) + cn \\ &\le bc\lfloor n/5 \rfloor + bc\lfloor 3n/4 \rfloor + cn \\ &\le bc(n/5) + bc(3n/4) + cn \\ &= (19/20)bcn + cn \end{aligned} \tag{1}$$

In order for $T(n) \le bcn$, we should have $(19/20)bcn + cn \le bcn$, or $b \ge 20$.

**(Q 11)** A graph $G = (V, E)$ consists of a vertex set $V$, with $|V| = n$ and an edge set $E$, and a relation that associates with each edge $e \in E$ two vertices $v_1, v_2 \in V$, $v_1$, $v_2$ not necessarily distinct. A simple graph is one where there are no self loops and multiple edges. Find out the following:

(i) How many graphs are possible with $n$ vertices? Graphs need not be simple.

(ii) How many graphs are possible with $n$ vertices with no vertices of degree $0$? Graphs need not be simple.

(iii) How many simple graphs are possible on $n$ vertices? [2+5+2=9]

**(Ans 11(i):)** The maximum number of edges possible in the graph is $|V| \times |V|$, i.e. $n^2$. An edge may either be present or not in the graph, making the total number of possible graphs to be $2^{n^2}$.

Also, any subset of $V \times V$ is a possible graph. The possible number of subsets of $V \times V$, and hence, the possible number of graphs is $2^{|V| \times |V|} = 2^{n^2}$.

**(Ans 11(ii):)** Let $A_i$ be the number of graphs possible $i$ $(1 \leq i \leq n)$ number of vertices of degree $0$. So, the number of graphs possible with $n$ vertices with no vertices of degree $0$ is $2^{n^2} - |\bigcup_{i=1}^{n} A_i|$. We now need to find $|\bigcup_{i=1}^{n} A_i|$.

To find $\sum_i |A_i|$, $(1 \leq i \leq n)$, we need to choose 1 vertex from $n$ vertices (in $\binom{n}{1}$ ways) that can serve as degree $0$ vertices, the rest $(n-1)$ vertices can form $2^{(n-1)^2}$ graphs. Thus, $\sum_i |A_i| = \binom{n}{1} 2^{(n-1)^2}$.

In counting $\sum_i |A_i|$, we included the case where we had two vertices with degree $0$. So, we need to compute $\sum_{j<i} |A_i \cap A_j|$. This is the number of graphs on $n-2$ vertices which is $2^{(n-2)^2}$; coupled with $\binom{n}{2}$ choices of $i$ and $j$, we have $\sum_{j<i} |A_i \cap A_j| = \binom{n}{2} 2^{(n-2)^2}$. Continuing this way, we have the case where all vertices are of degree $0$ as $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_n} = \binom{n}{n} 2^{(n-n)^2}$.

Thus, the final answer is

$$\begin{aligned} 2^{n^2} - \left| \bigcup_{i=1}^{n} A_i \right| &= 2^{n^2} - \left( \binom{n}{1} 2^{(n-1)^2} - \binom{n}{2} 2^{(n-2)^2} + \ldots + (-1)^{n-1} \binom{n}{n} 2^{(n-n)^2} \right) \\ &= \sum_{i=0}^{n} (-1)^i \binom{n}{i} 2^{(n-i)^2} \end{aligned}$$

**(Ans 11(iii):)** For a simple graph, the maximum number of edges possible is $\binom{n}{2}$. So, the total number of possible graphs is $2^{\binom{n}{2}}$.

**(Q 12)** Find the number of ways to build a nondecreasing wall using $n$ bricks. [5]

**(Ans 12:)** This is nothing but an unordered partition of a natural number $n$. That is, we are interested in counting the number of ways $p(n)$ to write a natural number as a sum of several natural numbers.

Let $x_i$ denote the number of bricks in columns of height exactly $i$ in the non-decreasing wall. We can use $n$ bricks in all. That leads us to the equation

$$x_1 + x_2 + \ldots + x_n = n.$$

with $x_1 \in \{0, 1, 2, 3, \ldots\}$, $x_2 \in \{0, 2, 4, 6, \ldots\}$, …, $x_i \in \{0, i, 2i, 3i, \ldots\}$. So, using generating functions, $p(n)$ is now the coefficient of $x^n$ in the product

$$\begin{aligned} G(x) &= (1 + x + x^2 + x^3 + \ldots) \cdot (1 + x^2 + x^4 + x^6 + \ldots) \cdots (1 + x^n + x^{2n} + x^{3n} + \ldots) \\ &= (1 - x)^{-1} \cdot (1 - x^2)^{-1} \cdots (1 - x^n)^{-1} \\ &= \prod_{i=1}^{n} \frac{1}{1 - x^i} \end{aligned}$$