Discrete Mathematics: Lectures 8 and 9
Principle of Inclusion and Exclusion

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As you can observe by now, we can count in various ways. One such method is the age-old *principle of inclusion and exclusion* that you have been dealing with possibly from your high school days. But again, have you ever asked as to why the cumbersome formula of principle of inclusion and exclusion is correct? We will look into it and also into some applications of the principle.

1 The formula for principle of inclusion and exclusion

Principle of inclusion and exclusion deals with the cardinality of finite sets. Let A_1 and A_2 be two finite sets. Then, we can easily show that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2|$. This is true because we first add up all the elements in the set A_1 and A_2; in doing so we have over counted those elements that were both in A_1 and A_2. So, we subtract once those elements to get the above formula. Taking this formula as the basis ($n = 2$) and using the distributive law that states for finite sets X, Y and Z, we have $X \cap (Y \cup Z) = (X \cap Y) \cup (X \cap Z)$, can you prove by induction the formula for the cardinality of the union of n finite sets $A_1, A_2, \ldots, A_n$?

Exercise 1 *Let $A_1, A_2, \ldots, A_n$ be n finite sets. Prove the following formula using induction.*

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots A_n| &= \sum_{i=1}^{n} |A_i| \\ &\quad - \sum_{1 \le i < j \le n} |A_i \cap A_j| \\ &\quad + \sum_{1 \le i < j < k \le n} |A_i \cap A_j \cap A_k| \\ &\quad - \ldots \\ &\quad + (-1)^{n-1} |A_1 \cap A_2 \cap \ldots A_n|. \end{aligned} \tag{1}$$

[Hints: The induction would be on n, the number of sets.]

We can have a series of inequalities corresponding to the above formula of principle of inclusion and exclusion.

$$|A_1 \cup A_2 \cup \ldots A_n| \le \sum_{i=1}^{n} |A_i| \tag{2}$$

$$\begin{aligned} |A_1 \cup A_2 \cup \ldots A_n| &\ge \sum_{i=1}^{n} |A_i| \\ &\quad - \sum_{1 \le i < j \le n} |A_i \cap A_j| \end{aligned} \tag{3}$$

We can have a series of such inequalities bounding $|A_1 \cup A_2 \cup \ldots A_n|$ from above and below till we include the last term $|A_1 \cap A_2 \cap \ldots A_n|$. These inequalities are useful when we have partial information about the sets and their intersections.

2 An alternative proof for principle of inclusion and exclusion

We look at a proof for the principle of inclusion and exclusion by counting. For this proof, we look at a notation that is different from Equation 1.

Observation 1 *As earlier, let* A_1, A_2, $\ldots$, A_n *be* n *finite sets. Then,*

$$\left|\bigcup_{i=1}^{n} A_i\right| \quad = \quad \sum_{j=1}^{n}(-1)^{j-1} \sum_{I \in \binom{\{1,2,\ldots,n\}}{j}} \left|\bigcap_{i \in I} A_i\right|. \tag{4}$$

Proof: Consider an arbitrary element $a \in |\bigcup_{i=1}^{n} A_i|$. a contributes exactly 1 to the left hand side of Equation 4. Let us now look into how much a contributes to the expression in the right hand side of Equation 4. Let a occur in exactly m ($m \leq n$) of the sets among the n sets A_1, A_2, $\ldots$, A_n. Let the m sets be A_{i_1}, A_{i_2}, $\ldots$, A_{i_m} where $1 \leq i_1 < i_2 < \ldots < i_m \leq n$. The element a now appears in every j-tuple ($1 \leq j \leq m$) of the sets among A_{i_1}, A_{i_2}, $\ldots$, A_{i_m} and in no other intersections. Varying j from 1 to m, we look at how many times the occurrence of a has been added. When $j = 1$, the occurrence of a has been added $\binom{m}{1}$ times with a positive sign as $(-1)^{1-1} = +1$. When $j = 2$, the occurrence of a has been added $\binom{m}{2}$ times with a negative sign as $(-1)^{2-1} = -1$. When $j = 3$, the occurrence of a has been added $\binom{m}{3}$ times with a positive sign as $(-1)^{3-1} = +1$. Continuing thus, when $j = m$, the occurrence of a has been added $\binom{m}{m}$ times with a sign equal to $(-1)^{m-1} = +1$. So, a contributes the following sum to the right hand side of the Equation 4

$$\binom{m}{1} - \binom{m}{2} + \binom{m}{3} - \ldots + (-1)^{m-1}\binom{m}{m} \tag{5}$$

We know from binomial theorem, that for any non-negative integer m, we have

$$(1+x)^m = \sum_{j=0}^{m} \binom{m}{j} x^j \tag{6}$$

Substituting $x = -1$ in Equation 6, we have

$$\begin{aligned} \binom{m}{0} - \binom{m}{1} + \binom{m}{2} - \binom{m}{3} + \ldots + (-1)^m \binom{m}{m} \quad &= \quad 0 \\ \binom{m}{1} - \binom{m}{2} + \binom{m}{3} + \ldots \text{ upto } m \text{ terms} \quad &= \quad 1 \left(= \binom{m}{0}\right) \end{aligned} \tag{7}$$

Thus, the Expression 5 contributes 1 to the right hand side of Equation 4. So, the contribution of any element $a \in |\bigcup_{i=1}^{n} A_i|$ is 1 to both sides of Equation 4 and hence, the formula holds. ■

In the remaining part, we look into several interesting applications of the principle of inclusion and exclusion.

3 Counting the number of onto functions

Let A and B be two sets with $|A| = n$ and $|B| = k$ with $n \geq k$. We want to count the number of onto functions from A to B. Consider the elements of set A to be n different balls and the elements of set B to be k different boxes. The problem of finding the number of onto functions from A to B is nothing but finding the number of ways that n different balls can be distributed among k different boxes with no box empty of balls. This condition of having no boxes empty basically enforces the onto function condition that for each $b \in B$ there exists $a \in A$ satisfying $f(a) = b$, i.e. for each box there is at least one ball.

The number of ways in which the n different balls can be distributed among the k different boxes with no restriction is nothing but the number of functions from A to B as each ball has to go into a box. So, the number of ways is $|B|^{|A|} = k^n$. From this, if we want to find the number of ways in which n different balls can be distributed among k different boxes with no box empty of balls, then we need to subtract the cases where any box i, $1 \leq i \leq k$, is empty; but in doing so we have subtracted the cases where two boxes can be empty. If we add the cases where two boxes can be empty, then we have added the cases where three boxes can be empty. This nature of the problem indicates that the counting can be done using principle of inclusion and exclusion.

Let A_i be the subset of the distributions that has the box i $(1 \leq i \leq k)$ empty; i.e. the set of functions that do not map to the element $i \in B$. So, we need to count the number of functions $f : A \rightarrow B \setminus \{i\}$, which is $(k-1)^n$. There are $\binom{k}{1}$ choices of i; so in all the number of such functions that do not map to an element i or the subset of the distributions that has the box i $(1 \leq i \leq k)$ empty is $\sum_i |A_i| = \binom{k}{1}(k-1)^n$. In counting $\sum_i |A_i|$, we included the case where we had two boxes empty. So, we need to compute $\sum_{j<i} |A_i \cap A_j|$. This is the number of functions $f : A \rightarrow B \setminus \{i, j\}$, which is $(k-2)^n$; coupled with $\binom{k}{2}$ choices of i and j, we have $\sum_{j<i} |A_i \cap A_j| = \binom{k}{2}(k-2)^n$. Continuing this way, we have the case where all the boxes are empty as $A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k} = \binom{k}{k}(k-k)^n$.

So, to find the total number of onto functions, we have to subtract the sum total of the cases where at least one box was empty ($|\cup A_i|$) from the total number of functions (k^n). Thus, the total number of onto functions is

$$k^n - |\cup A_i| \quad = \quad k^n - \left(\binom{k}{1}(k-1)^n - \binom{k}{2}(k-2)^n + \ldots + (-1)^{k-1}\binom{k}{k}(k-k)^n \right)$$

$$= \sum_{j=0}^{k}(-1)^j\binom{k}{j}(k-j)^n \tag{8}$$

$$\tag{9}$$

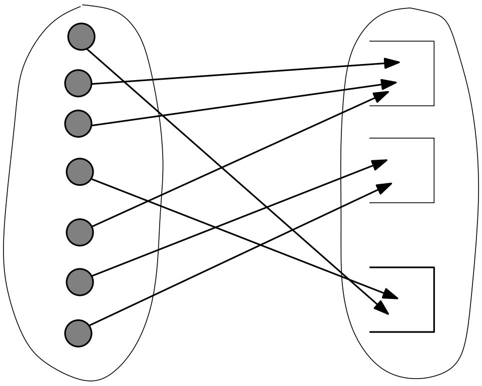

Figure 1: Distribution of balls into boxes.

4 Derangement

On a rainy day, n people, each with an umbrella, get into a pub to have a health drink. As the pub owner would not allow the soaked umbrellas to be taken inside, each person deposits his or her umbrella at the counter. While going out, the man at the counter returns the umbrellas at random. What is the probability that none of the persons gets back his or her own umbrella? We can have a similar problem with letters and envelopes where each letter has its designated envelope. Again, what is the probability that none of the letters goes into its designated envelope? This problem is also known as the *hatcheck problem* where umbrellas are replaced with hats! This class of problems where no item is going into its correct place is known as *derangement*. To fix the probability, we need to know the favourable cases. The possible number of cases is $n!$, i.e. all permutations.

4.1 Counting derangements

To calculate the number of favourable cases, we need to find out the number of permutations where none of the i items is going into the i-th position. For this, we introduce the concept of a *fixed point* of a permutation. Let Π denote a permutation, and $\Pi(i)$ denote the position of i in the permutation. Let $\{4, 2, 1, 3\}$ be a permutation of $\{1, 2, 3, 4\}$. Here, $\Pi(1) = 3$, $\Pi(2) = 2$, $\Pi(3) = 4$ and $\Pi(4) = 1$. We define a *fixed point* of a permutation as an index i for which $\Pi(i) = i$. So, to count the number of derangements we have to find out those permutations Π for which $\Pi(i) \neq i$ holds $\forall i \in \{1, 2, \ldots, n\}$. Now, if we can count the number of permutations that are 'bad' for derangement then we can subtract it from $n!$. A permutation is 'bad' for derangement if it has at least one fixed point. Let Π_n denote the set of all permutations of $\{1, 2, \ldots, n\}$ and for $i = 1, 2, \ldots, n$, we define a set A_i as the set of all permutations that has the index i as the fixed point, i.e. $A_i = \{\Pi \in \Pi_n | \Pi(i) = i\}$. The 'bad' permutations for derangement are those permutations that belong to $|\cup_i A_i|$. We have to now fix terms like $\sum_i |A_i|$, $\sum_{j<i} |A_i \cap A_j|$, $\sum_{k<j<i} |A_i \cap A_j \cap A_k|$, ..., $|A_1 \cap A_2 \cap \ldots \cap A_n|$. Each A_i is of size $(n-1)!$ and there are $\binom{n}{1}$ of them. Each $A_i \cap A_j$ is of size $(n-2)!$ and there are $\binom{n}{2}$ of them. Thus, we have

$$\begin{aligned} |\cup_{i=1}^{n} A_i| &= \binom{n}{1}(n-1)! - \binom{n}{2}(n-2)! + \ldots + (-1)^{n-1}\binom{n}{n}(n-n)! \\ &= \sum_{k=1}^{n} (-1)^{k-1} \binom{n}{k} (n-k)! \qquad (10) \end{aligned}$$

Thus, the number of derangements $D(n)$ is

$$\begin{aligned} D(n) &= n! - |\cup_{i=1}^{n} A_i| \\ &= n!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \ldots + (-1)^n \frac{1}{n!}\right) \qquad (11) \end{aligned}$$

So, the probability of a derangement is $\frac{D(n)}{n!}$. Notice that, as $n \to \infty$, this probability converges to e^{-1}, i.e. asymptotically the probability is independent of n.

4.2 Counting derangements in another way

As we have some idea about counting in terms of subproblems, let us see if we can apply it to derangement. Say, while computing $D(n)$, we know $D(n-1)$ and $D(n-2)$. Can we express $D(n)$ in terms of $D(n-1)$ and $D(n-2)$? Suppose, as earlier, we have n elements $\{1, 2, \ldots, n\}$. We have $n-1$ choices for the element i which appears in the first spot. For each of these, we either choose an arrangement with 1 in the i-th spot which we can do in $D(n-2)$ ways or choose an arrangement with 1 not in the i-th spot in $D(n-1)$ ways. Choosing an arrangement with 1 not in the i-th spot is equivalent to fixing i at the position 1 and as 1 cannot go to the i-th spot, we set the *fixed point* of 1 as i. Thus, we get $D(n-1)$ ways. See Figure 2. Thus, with $n-1$ choices of i, we finally have the recurrence for $D(n)$ as follows

$$\begin{aligned} D(n) &= (n-1)(D(n-1) + D(n-2)), n \geq 3 \\ &= 1, n = 2 \\ &= 0, n = 1 \end{aligned} \qquad (12)$$

References

[1] Fred S. Roberts and Barry Tesman, *Applied Combinatorics*, 2nd edition, Pearson, Prentice Hall, 2005, USA.

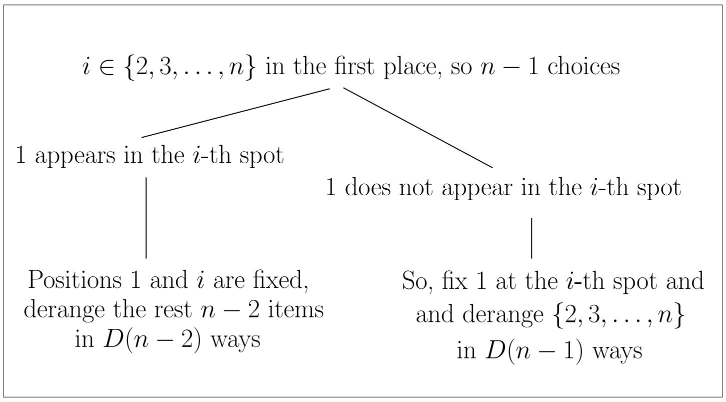

Figure 2: Explanation of the derangement recurrence.

[2] J. Matoušek and J. Nešetřil, *Invitation to Discrete Mathematics*, Oxford University Press, New York, 1998.

[3] C. L. Liu, *Elements of Discrete Mathematics*, Tata McGraw Hill, New Delhi, 2000.

[4] Ronald L. Graham, Donald E. Knuth and O. Patashnik, *Concrete Mathematics*, Pearson Education,