INDIAN STATISTICAL INSTITUTE

Assignment Set II

M. Tech (CS) - I Year, 2013-2014 (Semester - I)

Probability and Stochastic Processes

Deadline for submission: 13.09.2013 Total Marks : 170

Note: Answer all the problems legibly on fullscape paper. Clearly mark out your name, course, semester and roll number.

The assignments are given to aid your understanding of the subject. So, it is desired that you solve them yourself. In case you have taken any help, acknowledge it properly. Any instance of malpractice would be dealt with appropriately.

(Q1) In a chess board, a piece moves from $a1$ to $h8$, where $a, b, \ldots, h$ are the rows and $1, 2, \ldots, 8$ are the columns. It can either take one step forward horizontally or one step forward vertically. What is the probability that it goes through d5? [5]

(Q2) In how many different ways can eight identical cookies be distributed among three distinct children if each child receives at least two cookies and no more than four cookies? [5]

(Q3) A computer system considers a string of decimal digits a valid codeword if it contains an even number of 0 digits. For instance, 1230407869 is valid, whereas 120987045608 is not valid. Let a_n be the number of valid n-digit codewords. Find an explicit formula for a_n. [5]

(Q4) Computer processors are shipped in lots of 70 from a factory. Before being shipped, 25 are randomly tested from each lot. If any of these 25 fail, the entire lot is not shipped. What is the probability that a lot containing exactly 4 bad processors gets shipped? [5]

(Q5) The Census Bureau has estimated the following survival probabilities for men:

(a) probability that a man lives at least 70 years: 80%;

(b) probability that a man lives at least 80 years: 50%.

What is the conditional probability that a man lives at least 80 years given that he has just celebrated his 70th birthday? [5]

(Q6) In a population of children, 60% are vaccinated against whooping cough. The probabilities of contracting whooping cough are $\frac{1}{1000}$ if the child is vaccinated and $\frac{1}{100}$ if not. Find the probability that a child selected at random will contract whooping cough. [5]

(Q7) For the whooping cough exercise above, find the probability that a child is vaccinated given the occurrence of whooping cough. [5]

(Q8) What is the probability that at least 2 people in a random group of 6 people have a birthday on the same day of the week? [5]

(Q9) Two fair dies are thrown and their sum is observed. This is done repeatedly. What is the probability that a run of n consecutive 5s occurs before a run of m consecutive 7s? [5]

(Q10) Bob and Sue are in a group of 10 people who are seated randomly around a table. What is the probability that Bob and Sue are seated next to each other? [5]

(Q11) There are 40 different versions of an online homework. These are assigned randomly to the students. If a group of 3 classmates decide to work on the homework together, what is the probability that at least two of them receive the same version? [5]

(Q12) It is known that a student who does his homework on a regular basis has a chance of 83% to get a good grade; but the chance drops to 58% if he does not do the homework regularly. A particular student has been very busy with other courses and an evening job and finds that he has only a 69 percent chance of doing the homework regularly. What is his chance of not getting a good grade in the course? [5]

(Q13) Six cups and saucers come in pairs. There are two cups and saucers which are red, two white and two with stars on. If the cups and saucers are paired randomly, what is the probability that no cup is on a saucer of the same pattern? [5]

(Q14) Let A_r, $r \geq 1$ be events such that $P(A_r) = 1$ for all r. Show that $P\left(\bigcap_{r=1}^{\infty} A_r\right) = 1$. [5]

(Q15) Given that at least one but no more than three of the events A_r, $1 \leq r \leq n$, occur where $n \geq 3$. The probability that atleast two occur is $\frac{1}{2}$. If $P(A_r) = p$, $P(A_r \cap A_s) = q$ for $r \neq s$, and $P(A_r \cap A_s \cap A_t) = x$ for $r < s < t$, show that $p \geq \frac{3}{2n}$, and $q \leq \frac{4}{n}$. [5]

(Q16) A man has 5 coins, two double headed, one double tailed and two normal. He shuts his eyes, picks a coin at random and tosses it.

(a) What is the probability that the lower face of the coin is head?

(b) Now he opens his eyes and sees that the coin is showing heads. What is the probability that the lower face of the coin is head?

(c) Now he shuts his eyes and tosses the coin again. What is the probability that the lower face of the coin is head?

(d) Now he opens his eyes and sees that the coin is showing heads. What is the probability that the lower face of the coin is head?

(e) He discards this coin picks another at random and tosses it. What is the probability that this coin shows head?

[15]

(Q17) There are n urns. The rth urn contains $r - 1$ red balls and $n - r$ magenta balls. Pick an urn at random and remove two balls at random without replacement. Find the probability that

(a) the second ball is magenta,

(b) the second ball is magenta given that the first ball is magenta.

[5]

(Q18) Show that the conditional independence of A and B given C neither implies nor is implied by the independence of A and B. For which events C is it the case that, for all events A and B, A and B are independent iff they are conditionally independent given C? [5]

(Q19) Give an example where a set of events is pairwise independent but not mutually independent. [5]

(Q20) In a game show, the contestant is asked to select one of three doors. Behind two of the doors are goats and one of the doors hides a car. Success is selecting the door with the car behind it. After the contestant selects a door, the host opens one of the other two doors revealing a goat and asks if the contestant wishes to change his answer. Should the contestant change his answer? Justify your answer. [5]

(Q21) Each member of a group of n players rolls a die.

(a) For any pair of players who score the same number, the group scores 1 point. Find the mean and variance of the total score.

(b) For any pair of players who score the same number, the group scores that number. Find the mean and variance of the total score.

[5]

(Q22) Suppose a bookmaker offers payoff odds of $\pi(k)$ against the kth horse in an n horse race, where $\sum_{k=1}^{n} \{\pi(k) + 1\}^{-1} < 1$. Show that you can distribute your bets such that you always win. [5]

(Q23) A population of b animals has had a number a of its members captured, marked and released. Let X be the number of animals it is necessary to recapture (without re-release) in order to obtain m marked animals. Find $P(X = n)$ and $E[X]$. [5]

(Q24) Let X and Y be independent random variables, each taking value -1 and $+1$ with probability $\frac{1}{2}$. Let $Z = XY$. Are X, Y and Z mutually independent? Are they pairwise independent? [5]

(Q25) In your pocket is a random number N of coins, where $P(N = k) = e^{-\lambda}\frac{\lambda^k}{k!}$ for $k = 0, 1, 2, \ldots$. You toss each coin once, with probability p of showing heads. What is the distribution for the total number of heads? [5]

(Q26) Show that the variance of a geometric random variable with parameter p is $\frac{(1-p)}{p^2}$. [10]

(Q27) Consider the coupon collector's problem. Let X be the random variable that denotes the number of items to be bought to ensure n different coupons. Deduce that $E[X] = nH_n$, where H_n is the n-th Harmonic number. Compute $\text{Var}[X]$ and show that $\text{Var}[X] \leq \frac{\pi^2 n^2}{6}$. [10]

(Q28) Apply Markov's and Chebyshev's inequality to the coupon collector's problem to find out an upper bound on $P(X \geq rnH_n)$, where $r > 1$ and other notations are as in the previous problem. [15]