INDIAN STATISTICAL INSTITUTE

Assignment Set III

M. Tech (CS) - I Year, 2013-2014 (Semester - I)

Probability and Stochastic Processes

Deadline for submission: 29.11.2013 Total Marks : 80

Note: Answer all the problems legibly on fullscape paper. Clearly mark out your name, course, semester and roll number.

The assignments are given to aid your understanding of the subject. So, it is desired that you solve them yourself. In case you have taken any help, acknowledge it properly. Any instance of malpractice would be dealt with appropriately.

(Q1) In your pockets there is a random number N of coins, where N follows Poisson distribution with parameter λ. You toss each coin once with heads showing up eith probability p. What is the probability that k number of heads turn up? [5]

(Q2) A company has made a new product. Its demand is denoted by Poisson random variable X. The company must manufacture the units of the product in advance. A sold unit makes a profit b and an unsold unit incurs a loss c. How many units must be made to maximize expected profits? [5]

(Q3) Show that sum of n independent and identically distributed random variables has negative binomial distribution. [5]

(Q4) Consider a deck of cards. Cutting for the deal means each player removes a portion of the deck shows the bottom card of his segment. The highest card wins, ties not allowed. (Set the court cards as J=11, Q=12, K=13 and A=14 when aces high, A=1, otherwise.)

(a) M and N cut for the deal, aces high. Let X be M's card and Y be N's card. Find the joint pmf for X and Y

(b) Let V denote the loser's card and W denote the winner's card. Find the joint and marginal pmf's.

(c) Find the pmf of $W - V$.

(d) Find pmf of winner's card when three players are playing.

[10]

(Q5) You are given a fair coin and an arbitrary $\lambda \in [0,1]$. Use only this coin to form a game between two players A and B such that the probability of winning for A is λ. [5]

(Hint: Look at the binary expansion of λ.)

(Q6) $\sigma, \tau > 0$, $f(x,y) = \frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y}{\tau}-\frac{\rho x}{\sigma}\right)^2+\frac{x^2}{\sigma^2}-\frac{\rho^2 x^2}{\sigma^2}\right)\right]} dy$ for all $(x,y) \in \mathbb{R}^2$. Show that for $|\rho| < 1$, f is a pdf. Find the marginal pdfs of X and Y. [5]

(Q7) Let (X,Y) be uniformly distributed over the ellipse with boundary

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

of area $|C|$. What is $P(X > Y, X > -Y)$? [5]

(Hint: Consider changing to polar co-ordinates)

(Q8) Let X and Y be independent standard normal random variable. Find the pdfs of $X^2 + Y^2$ and $\tan^{-1}\left(\frac{Y}{X}\right)$. Are they independent? [5]

(Q9) X and Y are i.i.d exponential random variable. with parameter λ. Find the pdf of X conditioned on $X + Y = v$. [5]

(Q10) Let the joint distribution of X and Y be

$$f(x,y) = \frac{1}{2\pi\sigma\tau\sqrt{(1-\rho^2)}} e^{\left[-\frac{1}{2(1-\rho^2)}\left(\left(\frac{y}{\tau}-\frac{\rho x}{\sigma}\right)^2+\frac{x^2}{\sigma^2}-\frac{\rho^2 x^2}{\sigma^2}\right)\right]} dy$$

for all $(x,y) \in \mathbb{R}^2$ where $\sigma, \tau > 0$ and $|\rho| < 1$.

(a) Find the pdf of X conditioned on $Y = y$.

(b) Find $E\left[e^{tXY}\right]$.
(Hint: $E\left[e^{tXY}|Y)\right] = M_{X|Y}(tY)$ where M_Z denotes the moment generating function of Z)

(c) (X_1, Y_1) and (X_2, Y_2) are independent and have density f. Find the pdf of $Z = (X_1, Y_1) + (X_2, Y_2)$.
(Hint: Use the result in (b) and a result from moment generating functions)

[10]

(Q11) Show that if variance of a random variable is zero then it is a constant with probability 1. [5]

(Q12) Let $\{N(t)\,|t \geq 0\}$ be a set of independent random variables. $N(0) = 0$ and $N(t)$ has the distribution poisson(λt). Find $cov(N(s), N(t))$. [5]

(Q13) Let X and Y be discrete independent random variable with same mean. What is wrong with

$$E\left(X\,|X+Y=z\,\right)=E\left(X\,|X=z-Y\,\right)=E\left(z-Y\right)=z-E\left(Y\right)$$

[5]

(Q14) Let X and Y be continuous random variables with correlation ρ. Show that $E\left(var\left(Y\,|X\,\right)\right)\leq\left(1-\rho\right)^{2}var\left(Y\right)$. [5]

(Hint: Try using Cauchy-Schwarz inequality)