# Probability and Stochastic Processes (2023-24)
# Problem Sheet 3

1. Let $X$ and $Y$ be independent, uniform random variables on $[0,1]$. Find the density function and distribution function for $X+Y$.

2. Let $X$ and $Y$ be independent, exponentially distributed random variables with parameter 1. Find the density function and distribution function for $X+Y$.

3. The joint density of $X$ and $Y$ are given by:

$$f_{X,Y}(x,y) = \begin{cases} xe^{-(x+y)} & x>0, y>0 \\ 0 & \text{otherwise} \end{cases}$$

   0 otherwise (a) Are $X$ and $Y$ independent? (b) What are $f_X(x)$ and $f_Y(y)$? (c) What is $P(X+Y \leq 2)$?

4. Suppose Alice and Bob agreed to meet between 12 and 1 for a date at Starbucks. Because of their busy schedules, neither of them is sure when they will arrive; they know that for each of them, the arrival time is uniformly distributed over the hour. So that neither of them has to wait too long, they agreed that each of them would wait exactly 15 minutes for the other to arrive, and then leave. What is the probability their date will be successful?

5. If $X_1, X_2, \ldots, X_n$ are independent exponentially distributed random variables with parameters $\theta_1, \theta_2$, $\ldots, \theta_n$ respectively, then $min(X_1, X_2, \ldots, X_n)$ is exponentially distributed with parameter $\sum_{i=1}^{n} \theta_i$.

6. You are waiting at a bus stop to catch a bus across town. There are actually $n$ different bus lines you can take, each following a different route. Which bus you decide to take will depend on which bus gets to the bus stop first. As long as you are waiting, the time you have to wait for a bus on the $i$th line is exponentially distributed with mean $\mu_i$ minutes. Once you get on a bus on the $i$th line, it will take you $t_i$ minutes to get across town. Design an algorithm for deciding- when a bus arrives–whether or not you should get on the bus, assuming your goal is to minimize the expected time to cross town. (Hint: You want to determine the set of buses that you want to take as soon as they arrive. There are $2^n$ possible sets, which is far too large for an efficient algorithm. Argue that you need only consider a small number of these sets.)

7. Let $U$ be an $Uniform[0,1]$ r.v and let $a<b$ be constants. Show that:

   - If $b>0$ then $bU \sim Uniform([0,b])$.
   - $a+U \sim Uniform([a,a+1])$
   - What function of $U$ is distributed as $Uniform([a,b])$
   - Show that $min(U, 1-U) \sim Uniform(0,1/2)$

8. Let $X_1, X_2, \ldots$ be independent random variables that are uniformly distributed over $[1, -1]$. Show that the sequence $Y_1, Y_2, \ldots$ converges in probability to some limit, and identify the limit, for each of the following cases:

   (a) $Y_n = \frac{X_n}{n}$

   (b) $Y_n = (X_n)^n$

   (c) $Y_n = X_1 \cdot X_2 \cdots X_n$

   (d) $Y_n = \max\{X_1, \ldots, X_n\}$

9. Let $X_1.Y_1, X_2.Y_2, \ldots$ be independent random variables, uniformly distributed in the unit interval $[0, 1]$, and let

$$W = \frac{(X_1 + ... + X_{16}) - (Y_1 + ... + Y_{16})}{16}$$

   Find a numerical approximation to the quantity

$$\mathbf{P}(|W - \mathbf{E}[W]| < 0.001)$$