# Probability theory 1

Class Test 1
B. Stat 1st year
Maximum marks: 20
**Time**: 1 hour

November 24, 2021, 11:30 A.M. - 12:30 P.M.

Scanned copy of the answerscript is to be uploaded on Google Classroom or sent by email to sukrit049@gmail.com by 12:45 P.M.

1. (**10 points**) If 20 identical balls are thrown at random into 13 distinct boxes, calculate the probability that each box gets an even number of balls.

   **Soln.:** If $\Omega$ is the sample space, then it is clear that

   $$\#\Omega = \binom{32}{12},$$

   and all the outcomes in $\Omega$ are equally likely. If $A$ is the event that each box gets an even number of balls, then

   $$\#A = \text{Coefficient of } x^{20} \text{ in } (1 + x^2 + x^4 + \ldots)^{13}.$$

   For $|x| < 1$,

   $$\begin{aligned}(1 + x^2 + x^4 + \ldots)^{13} &= (1 - x^2)^{-13} \\ &= 1 + 13x^2 + \frac{13 \cdot 14}{2} x^4 + \ldots \\ &= \sum_{i=0}^{\infty} \binom{12+i}{i} x^{2i}.\end{aligned}$$

   The coefficient of $x^{20}$ in the above expression is $\binom{22}{10}$ and hence

   $$P(A) = \frac{\binom{22}{10}}{\binom{32}{12}}.$$

   Since the balls are identical, another was of arriving at the above answer is forming 10 pairs out of the 20 balls, and then distribution these 10 pairs into the 13 boxes, which can be done in $\binom{22}{12}$ ways by introducing 12 partitions.

2. (**10 points**) An urn contains 5 white balls, 5 black balls and 10 red balls. Balls are drawn from the urn with replacement until at least one ball of each of the three colours is drawn. Calculate the expected number of draws required.

**Soln.:** Let $X$ denote the number of draws needed to see balls of all colours. Denote for all $i = 1, 2, 3, \ldots$,

$$\begin{aligned}
W_i &= \{\text{At least one white ball is obtained by draw number } i\}\ , \\
B_i &= \{\text{At least one black ball is obtained by draw number } i\}\ , \\
R_i &= \{\text{At least one red ball is obtained by draw number } i\}\ .
\end{aligned}$$

Then, for $i \in \mathbb{N}$,

$$\begin{aligned}
&P(X > i) \\
&= P\left(W_i^c \cup B_i^c \cup R_i^c\right) \\
&= P(W_i^c) + P(B_i^c) + P(R_i^c) - P(W_i^c \cap B_i^c) - P(W_i^c \cap R_i^c) \\
&\qquad - P(B_i^c \cap R_i^c) + P(W_i^c \cap B_i^c \cap R_i^c) \\
&= \left(\frac{15}{20}\right)^i + \left(\frac{15}{20}\right)^i + \left(\frac{10}{20}\right)^i - \left(\frac{10}{20}\right)^i - \left(\frac{5}{20}\right)^i - \left(\frac{5}{20}\right)^i \\
&= 2\left[\left(\frac{3}{4}\right)^i - \left(\frac{1}{4}\right)^i\right] .
\end{aligned}$$

Hence,

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{\infty} P(X \geq i) \\
&= 1 + \sum_{i=1}^{\infty} P(X > i) \\
&= 1 + 2\sum_{i=1}^{\infty} \left[\left(\frac{3}{4}\right)^i - \left(\frac{1}{4}\right)^i\right] \\
&= 1 + 2\left(\frac{3/4}{1/4} - \frac{1/4}{3/4}\right) \\
&= 1 + 2\left(3 - \frac{1}{3}\right) \\
&= \frac{19}{3} .
\end{aligned}$$