Probability theory 1

Maximum marks: 12
Total Marks: 20

5 January 2022, 11:30 A.M. - 12:30 P.M.

Scanned copy of the answerscript is to be sent by email to sukrit049@gmail.com by 12:45 P.M. For late submissions, marks will be deducted at the rate of 0.5 per minute.

Answer all the questions. Each question carries 10 marks.

1. Let $(S_n : n \geq 0)$ be a simple symmetric random walk starting from zero. For a fixed $n \geq 2$, set

$$M = \max\{S_0, S_1, \ldots, S_{2n}\}\,.$$

(a) **(5 marks)** Calculate

$$P(M \geq 1, S_{2n} = 0)\,.$$

(b) **(5 marks)** Calculate

$$P(M = 1, S_{2n} = 0)\,.$$

Soln.: (a) Reflection principle implies that

$$\begin{aligned} P(M \geq 1, S_{2n} = 0) &= P(S_{2n} = -2) \\ &= \binom{2n}{n+1} 4^{-n}\,. \end{aligned}$$

(b) Write

$$\begin{aligned} P(M = 1, S_{2n} = 0) &= P(M \geq 1, S_{2n} = 0) - P(M \geq 2, S_{2n} = 0) \\ \text{(Reflection principle)} &= P(S_{2n} = -2) - P(S_{2n} = -4) \\ &= 4^{-n}\left[\binom{2n}{n+1} - \binom{2n}{n+2}\right] \\ &= 4^{-n}\frac{(2n)!}{(n-1)!(n+2)!}\left[(n+2) - (n-1)\right] \\ &= 4^{-n}3\frac{(2n)!}{(n-1)!(n+2)!}\,. \end{aligned}$$

2. Let

$$p_n = \frac{1}{n(n+1)}\,, n \in \mathbb{N}\,.$$

(a) **(4 marks)** Show that

$$\sum_{n=1}^{\infty} p_n = 1\,.$$

(b) **(3 marks)** If X is a random variable taking values $1, 2, 3, \ldots$ with probabilities $p_1, p_2, p_3, \ldots$, respectively, then show that

$$P(X \ge n) = \frac{1}{n}\,, n \in \mathbb{N}\,.$$

(c) **(3 marks)** Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values $1, 2, 3, \ldots$ with probabilities $p_1, p_2, p_3, \ldots$, respectively. Set

$$Z_n = \#\{1 \le i \le n : X_i \ge n\}\,.$$

For $j = 0, 1, 2, 3, \ldots$, calculate

$$\lim_{n\to\infty} P(Z_n = j)\,.$$

Soln.: (a) The definition of infinite sums imply

$$\begin{aligned}
\sum_{n=1}^{\infty} p_n &= \lim_{n\to\infty} \sum_{i=1}^{n} p_i \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{i(i+1)} \\
&= \lim_{n\to\infty} \sum_{i=1}^{n} \left[\frac{1}{i} - \frac{1}{i+1}\right] \\
&= \lim_{n\to\infty} \left(1 - \frac{1}{n+1}\right) \\
&= 1\,.
\end{aligned}$$

(b) For $n \in \mathbb{N}$, similar arguments as in (a) imply that

$$P(X \ge n) = \sum_{i=n}^{\infty} \left[\frac{1}{i} - \frac{1}{i+1}\right] = \frac{1}{n}\,.$$

(c) From (b), it follows that Z_n follows $\mathrm{Bin}(n, 1/n)$. Hence, for $j \ge 0$,

$$\lim_{n\to\infty} P(Z_n = j) = P(Z = j)\,,$$

where Z follows Poisson(1). Thus, for $j = 0, 1, \ldots,$

$$\lim_{n\to\infty} P(Z_n = j) = \frac{e^{-1}}{j!}\,.$$