1. **(20 points)** Let $(S_n : n \geq 0)$ be an asymmetric random walk starting from 0, that is, it takes a positive step with probability p and a negative one with probability $1-p$ for some $0 < p < 1$ with $p \neq \frac{1}{2}$.

   (a) Prove that

$$\lim_{n\to\infty} \frac{P(S_{2n}=0)}{n^{-1/2}(4p(1-p))^n} = \frac{1}{\sqrt{\pi}}\,.$$

   (b) Let

$$V_n = \#\{1 \leq i \leq 2n : S_i = 0\}\,, n \geq 1\,.$$

   Show that

$$\sup_{n\geq 1} \mathrm{E}(V_n) < \infty\,.$$

Soln.: (a) Recall Stirling's approximation which says that

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n, n \to \infty\,.$$

Use the above to write

$$\begin{aligned} P(S_{2n}=0) &= \binom{2n}{n} p^n (1-p)^n \\ (\text{as } n\to\infty) &\sim \sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n} \frac{1}{2\pi n}\left(\frac{n}{e}\right)^{-2n} p^n(1-p)^n \\ &= \frac{1}{\sqrt{\pi n}}\left(4p(1-p)\right)^n\,. \end{aligned}$$

(b) Note that

$$\sum_{n=1}^{\infty} n^{-1/2}(4p(1-p))^n \leq \sum_{n=1}^{\infty}(4p(1-p))^n < \infty\,,$$

because $0 \leq 4p(1-p) < 1$ if $p \in [0,1]$ and $p \neq 1/2$. Using this in conjunction with (a), the ratio test shows that

$$\sum_{n=1}^{\infty} P(S_{2n}=0) < \infty\,.$$

Write

$$V_n = \sum_{i=1}^{n} \mathbf{1}(S_{2i}=0)\,,$$

to get that

$$\mathrm{E}(V_n) = \sum_{i=1}^{n} P(S_{2i}=0) \leq \sum_{i=1}^{\infty} P(S_{2i}=0)\,.$$

Thus,

$$\sup_{n\geq 1} \mathrm{E}(V_n) \leq \sum_{i=1}^{\infty} P(S_{2i}=0) < \infty\,.$$

2. **(20 points)** An urn contains 10 white balls, 15 black balls and 10 red balls. All the 35 balls are distinguishable. Balls are drawn from the urn without replacement until all the red balls are drawn. Calculate the expected number of white balls drawn.

Soln.: Label all the white balls as $W_1, \ldots, W_{10}$. Let X be the number of white balls drawn. Assume that balls are drawn from the urn, one by one, till the urn becomes empty. In that case X is same as the number of white balls drawn before the last red ball drawn. That is,

$$X = \sum_{i=1}^{10} \mathbf{1}\left(W_i \text{ is drawn before the last red ball is drawn}\right).$$

Therefore,

$$\begin{aligned}
\mathrm{E}(X) &= \sum_{i=1}^{10} P\left(W_i \text{ is drawn before the last red ball is drawn}\right) \\
&= \sum_{i=1}^{10} \left[1 - P\left(W_i \text{ is drawn after the last red ball is drawn}\right)\right] \\
&= \sum_{i=1}^{10} \Big[1 - P\Big(\text{in a random arrangement of 1 white ball and} \\
&\qquad \text{10 red balls, the white ball comes at the end}\Big)\Big] \\
&= 10\left(1 - \frac{1}{11}\right) \\
&= \frac{100}{11}.
\end{aligned}$$

3. **(20 points)** Consider a simple symmetric random walk $(S_n : n \geq 0)$ starting from 0. Define

$$T := \inf\{n \geq 1 : S_n = 0\} .$$

Calculate

$$\lim_{n\to\infty} n^{3/2} P(T = 2n) .$$

Soln.: It has been shown in Lecture 24 (December 14) that

$$P(S_1 \neq 0, \ldots, S_{2n} \neq 0) = P(S_{2n} = 0)\,, n \geq 1 . \tag{1}$$

Write for $n \geq 2$,

$$\begin{aligned} P(T = 2n) &= P(T \geq 2n) - P(T \geq 2n+2) \\ &= P(S_1 \neq 0, \ldots, S_{2n-2} \neq 0) - P(S_1 \neq 0, \ldots, S_{2n-2} \neq 0) \\ &= P(S_{2n-2} = 0) - P(S_{2n} = 0)\,, \end{aligned}$$

(1) being used in the last line. Thus,

$$\begin{aligned} P(T = 2n) &= \binom{2n-2}{n-1} 4^{-n+1} - \binom{2n}{n} 4^{-n} \\ &= 4^{-n}\left(4\binom{2n-2}{n-1} - \binom{2n}{n}\right) \\ &= 4^{-n}(2n-2)!\left(\frac{4}{((n-1)!)^2} - \frac{2n(2n-1)}{(n!)^2}\right) \\ &= 4^{-n}(2n-2)!(n!)^{-2}\left(4n^2 - 2n(2n-1)\right) \\ &= 4^{-n}2n(2n-2)!(n!)^{-2} \\ &= \frac{1}{4^n(2n-1)}(2n)!(n!)^{-2} . \end{aligned}$$

Stirling's approximation implies that as $n \to \infty$,

$$\begin{aligned} P(T = 2n) &\sim \frac{1}{4^n(2n-1)}\sqrt{2\pi 2n}\left(\frac{2n}{e}\right)^{2n} \frac{1}{2\pi n}\left(\frac{n}{e}\right)^{-2n} \\ &= \frac{1}{(2n-1)\sqrt{\pi n}} \\ &\sim \frac{1}{2\sqrt{\pi} n^{3/2}} . \end{aligned}$$

That is,

$$\lim_{n\to\infty} n^{3/2} P(T = 2n) = \frac{1}{2\sqrt{\pi}} .$$

4. **(15 points)** Suppose that for $n \geq 3$, $A_1, \ldots, A_n$ are pairwise independent events such that no three of them can occur simultaneously. If $P(A_i) = p$ for all $i = 1, \ldots, n$, calculate the probability that none of $A_1, \ldots, A_n$ occurs.

Soln.: The inclusion-exclusion principle implies that

$$\begin{aligned}
&P\left(\bigcap_{i=1}^{n} A_i^c\right) \\
&= 1 - P\left(\bigcup_{i=1}^{n} A_i\right) \\
&= 1 - \sum_{i=1}^{n} P(A_i) + \sum_{1\leq i<j\leq n} P(A_i \cap A_j) \\
&\quad - \sum_{1\leq i<j<k\leq n} P(A_i \cap A_j \cap A_k) + \ldots \\
&= 1 - \sum_{i=1}^{n} P(A_i) + \sum_{1\leq i<j\leq n} P(A_i \cap A_j) \\
&= 1 - np + \binom{n}{2} p^2 ,
\end{aligned}$$

the penultimate line following from the hypothesis that no three of $A_1, \ldots, A_n$ can occur simultaneously, and the last line follows from pairwise independence of $A_1, \ldots, A_n$.

Thus, probability that none of the events $A_1, \ldots, A_n$ occurs equals $1 - np + \binom{n}{2}p^2$.

5. **(15 points)** A particular kind of bacterium either splits into 3 bacteria of the same kind or dies, with probability p and $1-p$ respectively. At any point of time, all such bacteria that exist behave in the described way, independently. If a system contains 1 bacterium of this kind, find the probability that eventually the system will be free of such bacteria, assuming that bacteria from outside the system cannot enter it.

Soln.: The described system is a branching process with progeny distribution

$$p_k := \begin{cases} 1-p, & k=0\,, \\ p, & k=3\,, \\ 0, & \text{otherwise}\,. \end{cases}$$

The progeny PGF is

$$\phi(s) = q + ps^3\,,$$

where $q = 1-p$.

Solving $\phi(s) = s$, we get

$$\begin{aligned} 0 &= ps^3 - s + q \\ &= ps(s+1)(s-1) - q(s-1) \\ &= (s-1)(ps^2 + ps - q)\,. \end{aligned}$$

The roots of the above equation are 1 and $(-p \pm \sqrt{p^2+4pq})/2p$. The root $(-p - \sqrt{p^2+4pq})/2p$ is negative, and hence can be outright rejected. Both the other roots are non-negative, and therefore, the extinction probability is

$$\begin{aligned} p_e &= \min\left\{(\sqrt{p^2+4pq} - p)/2p, 1\right\} \\ &= \begin{cases} 1, & 0 \le p \le \frac{1}{3}\,, \\ \frac{\sqrt{p^2+4pq}-p}{2p}, & \frac{1}{3} < p \le 1\,. \end{cases} \end{aligned}$$

6. **(15 points)** A fair die is rolled 20 times. What is the probability that the sum of the numbers obtained is strictly larger than 25?

Soln.: Let Ω denote the sample space, and A the event of interest. Clearly,

$$\#\Omega = 6^{20}\,,$$

and

$$\begin{aligned}
\#A^c &= \#\left\{(k_1,\ldots,k_{20}) : 1 \le k_i \le 6,\ \sum_{i=1}^{20} k_i \le 25\right\} \\
&= \#\left\{(k_1,\ldots,k_{20}) : 1 \le k_i,\ \sum_{i=1}^{20} k_i \le 25\right\} \\
&= \#\left\{(k_1,\ldots,k_{20}) : k_i \ge 0,\ \sum_{i=1}^{20} k_i \le 5\right\} \\
&= \#\left\{(k_1,\ldots,k_{21}) : k_i \ge 0,\ \sum_{i=1}^{21} k_i = 5\right\} \\
&= \binom{25}{5}\,.
\end{aligned}$$

Therefore,

$$P(A) = 1 - 6^{-20}\binom{25}{5}\,.$$