Probability theory 1

Homework 1

September 29, 2021

These problems are for your own practice, and are not to be submitted. Please discuss among yourselves and make sure that you understand the solution. For a good understanding, you may want to write down the solution. These will be discussed in a tutorial session.

1. A coin with chances of Heads p is tossed until the first Tail is obtained. Calculate the probability that the number of tosses required is an odd number.

2. A fair die is rolled 9 times. Calculate the probability that the sum of the numbers obtained equals 15.

3. A fair coin is tossed n times. Calculate the probability that the number of Tails obtained is divisible by 4.

4. If 4 cards are chosen at random from a standard deck, calculate the probability that at least 2 cards will have the same rank.

5. If the 26 letters of the alphabet are randomly permuted, calculate the probability that vowels occur next to each other.

6. An urn contains 5 black balls, 5 white balls and 10 red balls, all the 20 balls being distinct. Seven balls are chosen from the urn without replacement. Calculate the probability that at least one ball of each colour is drawn. What if the balls were chosen with replacement?

7. Each of a hundred sticks is broken into two parts - a long part and a short part. Each long part is randomly joined with a short part. What is the probability that we get back at least one original stick?

8. Ten identical balls are thrown at random into seven distinguishable boxes. What is the probability that no box gets more than two balls?

9. Ten distinct balls are thrown at random into one of three distinct boxes, at random. Find the probability that at least one of the boxes is empty.

10. Two rooks of different colours are placed at random on an empty chess board. What is the probability that they are in positions such that they can take each other?

11. If n distinguishable balls are placed at random into n distinguishable boxes, find the probability that exactly one box remains empty.

12. A box contains N tickets numbered $1, 2, \ldots, N$. From this box n tickets are drawn without replacement. What is the probability that the largest number drawn is M? What if the tickets were drawn with replacement?

13. From the set of all possible functions $f : \{1, \ldots, m\} \to \{1, \ldots, n\}$, one is chosen at random.

    (a) Assuming that $m \leq n$, calculate the probability that the chosen function is one-one.

    (b) When $m = n$, calculate the probability that the chosen function is a bijection, that is, it is one-one and onto.

14. A number is chosen at random from the first n natural numbers. For a fixed number k, calculate the probability p_n that the chosen number is divisible by k. Evaluate the following limit:

$$\lim_{n \to \infty} p_n \,.$$

15. Three fair dice are rolled.

    (a) What is the probability that the same number appears on at least two of the dice?

    (b) What is the probability that the same number appears on exactly two of the dice?