Probability theory 1

Homework 2

October 9, 2021

1. Consider a disease which 1% of the population has contracted. A diagnostic test for the disease gives the correct result with probability 0.9, independently of whether the subject being tested has the disease or not. If the test gives a positive result for a certain individual, what is the chance that he actually has the disease?

2. Consider a minor tweak of the above problem. A disease is contracted by a proportion p of the population. A diagnostic test for the disease yields a correct result with probability 0.9, independently of whether the subject being tested has the disease.

   (a) Evaluate
   $$\lim_{p\to 0} P\left(\text{The subject has the disease} \mid \text{he tests positive}\right) .$$
   Convince yourself that the rareness of the disease causes the above conditional probability to be low, which is why usually a subject is asked to undergo a second test if the first test yields a positive result.

   (b) Evaluate
   $$\lim_{p\to 0} P\left(\text{The subject has the disease} \mid \text{she tests negative}\right) .$$
   When someone tests negative for a rare disease, the doctors usually do not go for a second test. Does your answer justify this action of the doctors?

3. A box contains a white and b black balls. Balls are drawn one by one without replacement until balls of the same colour are left in the box. What is the probability that the balls left are all white?

4. Suppose that 5 percent of men and 2.5 percent of women are color-blind. A color-blind person is chosen at random. What is the probability that this person is male? Assume that the number of males and females are equal.

5. Two cards are chosen at random without replacement from a standard deck. What is the conditional probability that they have the same rank, given that their suits are different?

6. Suppose that A_1 and A_2 are events with $P(A_i) = p_i > 0$, for $i = 1, 2$. Calculate $P(A_1|A_2)$ assuming that

   (a) A_1 and A_2 are disjoint,

   (b) $A_1 \subset A_2$,

   (c) $A_1 \supset A_2$.

7. An examination centre has 50 rooms. In each room, 10 examinees have been assigned. Each examinee is absent with probability 0.2. The choice of being present or absent is made independently by each examinee. What is the probability that at least one room will have full attendance?

8. A number is chosen at random from $\{0, 1, \ldots, 10^n - 1\}$ for some $n \geq 1$. Define the event

$$A_i = \{\text{The } i\text{-th digit from right is an odd number}\},\ 1 \leq i \leq n\,.$$

   (a) Are $A_1, \ldots, A_n$ independent?

   (b) Calculate the probability that the product of all the n digits is even.

9. Consider the Polya urn scheme with w white balls and b black balls. Suppose that after each draw, c balls having the colour of the drawn ball are added to the urn. [For $c = 0$, this is same as drawing without replacement, and for $c = 1$ this is drawing with replacement.] Calculate the probability that the n-th draw is a white. Write down a complete proof of whatever you claim.

10. Suppose that $A_1, \ldots, A_n$ are events such that $P(A_1 \cap \ldots \cap A_n) > 0$.

    (a) Show that

$$P(A_1 \cap \ldots \cap A_n) =$$
$$P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2) \ldots P(A_n|A_1 \cap \ldots A_{n-1})\,.$$

    (b) Show that

$$P(A_1|A_2 \cap \ldots \cap A_n) = \frac{P(A_1 \cap A_n|A_2 \cap \ldots \cap A_{n-1})}{P(A_n|A_2 \cap \ldots \cap A_{n-1})}\,.$$

11. Let $A_1, \ldots, A_n$ be independent events.

    (a) Show that for any $2 \leq k \leq n$, $A_1, \ldots, A_k$ are independent.

(b) Show that for any $2 \leq k \leq n$, $A_1, \ldots, A_{k-1}, A_k^c$ are independent.

(c) Show that for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ and $1 \leq j_1 < j_2 < \ldots < j_l \leq n$ such that

$$\{i_1, \ldots, i_k\} \cap \{j_1, \ldots, j_l\} = \emptyset\,,$$

the events $A_{i_1}, \ldots, A_{i_k}, A_{j_1}^c, \ldots, A_{j_l}^c$ are independent.

12. Let A, B and C be events, with A and B independent and of positive probability. Show that

$$P(C|A) = P(C|A \cap B)P(B) + P(C|A \cap B^c)P(B^c)\,.$$

13. Suppose that the events $A_1, A_2, \ldots$ are disjoint, exhaustive and have positive probability. If for an event B it holds that

$$P(B|A_i) = P(B|A_j) \text{ for all } i, j\,,$$

show that B is independent of A_i for every i.

14. For two events A and B with probabilities strictly between 0 and 1, decide whether the following are true or false. If a claim doesn't hold in a **single** case, then it is false.

(a) $P(A^c|B) = 1 - P(A|B)$

(b) $P(A|B^c) = 1 - P(A|B)$

(c) $P(A|B^c) = P(A) - P(A|B)$

15. Suppose that the events $A_1, \ldots, A_n$ are pairwise independent, each occuring with probability p. Assume that no three of these events can simultaneously occur. Calculate the probability that none of the events occur.