Probability theory 1

Homework 3

November 19, 2021

1. If X is a random variable taking values $0, 1, 2, \ldots$, and having an expectation, then show that

$$\mathrm{E}(X) = \sum_{n=1}^{\infty} P(X \geq n)\,.$$

2. Suppose that n balls are thrown at random into n boxes. Let X_n denote the number of balls in Box 1. Calculate

$$\lim_{n\to\infty} P(X_n = k)\,,$$

for every $k = 0, 1, 2, \ldots$, etc.

3. Consider a society of n individuals. Assume that for every $1 \leq i < j \leq n$, the i-th individual is a friend of the j-th individual with probability n^{-2}. Let A_{ij} be the event that the i-th and the j-th individuals are friends. Assume that the collection of events $(A_{ij} : 1 \leq i < j \leq n)$ are independent. If X_n denotes the number of friendships in the society, calculate

$$\lim_{n\to\infty} P(X_n = k)\,,$$

for every $k = 0, 1, 2, \ldots$, etc.

4. Imagine n electric poles, located one after another. Whenever there is a storm, the connection between two consecutive poles is broken with probability λ/n, for some fixed $\lambda > 0$. Let p_n be the probability that a storm does no damage. Find

$$\lim_{n\to\infty} p_n\,.$$

5. A sample of size n is drawn with replacement from $\{1, \ldots, n\}$. Let X_n denote the minimum of the sample units. For $k = 1, 2, \ldots$, calculate

$$\lim_{n\to\infty} P(X_n = k)\,.$$

6. Suppose that n numbers are chosen with replacement from $\{1, \ldots, N\}$. Let X denote the minimum of the n chosen numbers.

   (a) Calculate the PMF of X and $\mathrm{E}(X)$.

   (b) Calculate the limit of $P(X = 1)$ and $\mathrm{E}(X)$ as $n \to \infty$.

7. I have a coin whose chance of Heads is p in a single toss. The coin is tossed until k Heads are obtained, where $k \geq 1$. Let X denote the number of tosses required. Calculate the PMF, expectation and variance of X. The distribution of X is called Negative binomial (k, p).

8. A box has M white balls and $N - M$ black balls. Balls are drawn without replacement from the box. Let X be the number of black balls before the appearance of a white ball. Find the distribution and expectation of X.

9. A die is rolled until the total of the points observed exceeds five. Let X denote the number of rolls required. Calculate the distribution and the expectation of X.

10. A man has a key ring with n keys, exactly one of which would unlock a door. He has forgotten the correct key, and keeps picking keys from the ring, with replacement, and tries them. If X denotes the number of trials needed to unlock the door, calculate $\mathrm{E}(X)$.

11. The wife of the man in the above problem tells him that a better strategy is to try the keys without replacement. Justify or contradict her claim by recalculating $\mathrm{E}(X)$ in this scheme, and comparing the same with your answer to the above problem.

12. Apply the following tweak to the above problem. Suppose that the ring has exactly two keys which would unlock the door. Recalculate the expected number of trials needed to unlock the door, when the keys are tried

    (a) with replacement,

    (b) without replacement.

13. A set is chosen at random from the collection of all subsets of $\{1, \ldots, n\}$. Let X denote the cardinality of the chosen set. Calculate the PMF and expectation of X.

14. For events $A_1, A_2, \ldots$, show that

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} P(A_n) .$$

15. Calculate the expectation and variance of the the following distributions. Even if some of these are done in class, redo the calculations for your own practice.

    (a) Bernoulli (p) (Bernoulli (p) is the distribution of a random variable taking values 1 and 0 with probabilities p and $1-p$, respectively.)

    (b) Binomial (n, p)

    (c) Geometric (p)

    (d) Poisson (λ)

    (e) Uniform on $\{1, \ldots, n\}$