Probability theory 1

Homework 4

November 30, 2021

1. You have a coin whose chance of Heads in a single toss is p. The coin is tossed until you get two Heads. Let X be the number of tosses needed to get the first Heads, and let Y be the number of tosses, after the first Heads, to get the second Heads. Calculate the joint PMF of X and Y. Are they independent?

2. Let X and Y follow Binomial (m, p) and Binomial (n, p), respectively, independently of each other. Find out the distribution of $X + Y$.

3. If X and Y are independent Poisson random variables having means λ and μ, respectively, find the distribution of $X + Y$.

4. Suppose that the random variable X is distributed uniformly on the set $\{-N, -N+1, \ldots, N\}$. Define

$$Y = X^2 .$$

   Calculate $\text{Cov}(X, Y)$. Are X and Y independent?

5. Let X and Y be independent random variables following Geometric(p_1) and Geometric(p_2) respectively. Calculate the PMF of

   (a) $X + Y$,

   (b) $\min\{X, Y\}$,

   (c) $\max\{X, Y\}$.

6. Suppose that n balls are thrown at random into n boxes. Let X_n denote the proportion of empty boxes. Show that

$$\lim_{n\to\infty} \mathrm{E}(X_n) = e^{-1} ,$$

   and

$$\lim_{n\to\infty} \mathrm{Var}(X_n) = 0 .$$

7. Let n identical balls be thrown into k boxes. Calculate the expectation and the variance of the number of empty boxes.

8. A card is selected at random from a set of 100 cards which are numbered $00, 01, \ldots, 99$. Let X and Y be respectively the sum and the product of the two digits on the selected card. Find the joint PMF of X and Y and $\mathrm{Cov}(X, Y)$.

9. Suppose that we have an infinite supply of balls and N boxes. Fix $k \leq N$. The balls are thrown at random into the boxes until exactly k boxes are occupied. Let X denote the number of throws needed for this. Calculate $\mathrm{E}(X)$ and $\mathrm{Var}(X)$.

10. For random variables X and Y with respective variances σ_X^2 and σ_Y^2, show that
$$(\sigma_X - \sigma_Y)^2 \leq \mathrm{Var}(X + Y) \leq (\sigma_X + \sigma_Y)^2 \,.$$

11. A sample of size n is drawn from $\{1, \ldots, N\}$ without replacement. Fix $1 \leq i < j \leq n$, and let X_i and X_j denote the indicators of the events that the i-th and the j-th units are at most $N/2$, respectively. Calculate the joint PMF of X_i and X_j.

12. A sample of size n is drawn from $\{1, \ldots, N\}$ with replacement. Let X denote the minimum of the numbers drawn. Calculate
    (a) the CDF of X,
    (b) the PMF of X,
    (c) $\mathrm{E}(X)$,
    (d) and $\mathrm{Var}(X)$.
    (e) If Y denotes the maximum of the numbers drawn, calculate the joint PMF of X and Y.

13. Let X and Y be independent Poisson(λ) random variables. Calculate $P(X < Y)$ and $P(X = Y)$.

14. Let $X_1, \ldots, X_n$ be random variables such that for all $i \neq j$, X_i and X_j are independent, and
$$\begin{aligned} \mathrm{E}(X_i) &= \mu_i \,, \\ \mathrm{Var}(X_i) &= \sigma_i^2 \,. \end{aligned}$$
Show that
$$\mathrm{E}\left[\left(\sum_{i=1}^n X_i\right)^2\right] = \sum_{i=1}^n \sigma_i^2 + \left(\sum_{i=1}^n \mu_i\right)^2 \,.$$

15. An urn contains W white balls and B black balls. From the urn, n balls are drawn without replacement, for some $1 \leq n \leq W + B$. Denoting by X the number of white balls drawn, find the PMF and expectation of X. The distribution of X is called Hypergeometric(W, B, n).