Probability theory 1

Homework 5

December 6, 2021

1. Let $X_1, \ldots, X_n$ have respective variances $\sigma_1^2, \ldots, \sigma_n^2$. Show that
$$\mathrm{Var}\left(\sum_{i=1}^n X_i\right) \le \left(\sum_{i=1}^n \sigma_i\right)^2 .$$

2. Let X follow an uniform distribution on $\{-N, -N+1, \ldots, N\}$. Define
$$Y = \mathbf{1}(X \ge 0)\,,$$
that is, Y takes the value 1 if $X \ge 0$, 0 otherwise. Calculate the correlation between X and Y.

3. Suppose that X follows Binomial$(9, 1/3)$ and Y follows Poisson(2). Nothing is told about the joint distribution of X and Y. Define
$$\begin{aligned} U &= X + Y\,, \\ V &= X - Y\,. \end{aligned}$$
Calculate the correlation between U and V.

4. Let X follow Poison(λ), and given $X = k$, for every $k = 0, 1, 2, \ldots$, the conditional distribution of Y is Binomial(k, p). Calculate the distribution of Y. Do the same problem, with the distribution of X replaced by Binomial(n, r) for some $n \ge 1$ and $0 \le r \le 1$.

5. A fair die is rolled till the first 6 is observed. Let X denote the number of rolls needed for this. Given that all the rolls, till the first 6, give even numbers, find the conditional expectation of X. Calculate the probability that all the rolls till the first 6 are even.

6. Once again a fair die is rolled till the first 6. Calculate the expectation of the sum of the numbers obtained on all the throws, including the one that gives the 6.

7. Suppose that X and Y follow Bin(n, p) and Bin(m, p), respectively and independently of each other. For every $k = 0, 1, \ldots$, calculate the conditional distribution of X given that $X + Y = k$.

8. A coin with probability of Heads being p is tossed repeatedly.

   (a) Calculate the expected number of tosses to get a Heads followed by a Tails in consecutive tosses.

   (b) Calculate the expected number of tosses to get two consecutive Heads.

9. Let X and Y be independent Poisson random variables with means λ and μ, respectively. Find the conditional distribution of X given that $X + Y = k$, for a fixed $k \geq 0$.

10. If X is a random variable having expectation and $P(X > 0) > 0$, then show that
$$\mathrm{E}(X|X > 0) > 0\,.$$

11. If X and Y are random variables such that for every $x \in \mathbb{R}$,
$$P(X = x|Y = y)$$
remains unchanged for every y such that $P(Y = y) > 0$, then show that X and Y are independent.

12. Let p be the joint PMF of $(X_1, \ldots, X_n)$. If $f : \mathbb{R}^n \to \mathbb{R}$ is such that $f(X_1, \ldots, X_n)$ has an expectation, then
$$\mathrm{E}\left[f(X_1, \ldots, X_n)\right] = \sum_{x \in \mathbb{R}^n : p(x) > 0} f(x)p(x)\,.$$

13. Let $X_1, \ldots, X_n$ be i.i.d. from a Binomial(k, p) distribution. Find the conditional distribution of X_1 given that $X_1 + \ldots + X_n = r$, for a fixed $r \in \{0, 1, \ldots, nk\}$.

14. Let $X_1, \ldots, X_n$ be i.i.d. from Geometric(p). Find the distribution of

   (a)
$$\min\left(X_1, \ldots, X_n\right)\,,$$

   (b)
$$\sum_{i=1}^{n} X_i\,.$$

15. Let $X_1, \ldots, X_n$ be independent random variables following Poisson with parameters $\lambda_1, \ldots, \lambda_n$, respectively. Find the distribution of
$$X_1 + \ldots + X_n\,.$$