Probability theory 1

Homework 6

December 14, 2021

1. Consider a symmetric random walk $(S_n : n \geq 0)$. Let T_n denote the number of visits to the origin till time n. Show that there exist $C, \beta \in (0, \infty)$ such that
$$\mathrm{E}(T_n) \sim Cn^{\beta} ,$$
as $n \to \infty$.

2. In the above random walk, let T be the first time of visit to the origin.
   (a) Calculate $P(T > 2n)$ for each $n \geq 1$.
   (b) Argue that sooner or later, the random walk will visit the origin, some time.
   (c) Does T have an expectation?

3. Let $(S_n : n \geq 0)$ be a symmetric random walk. Fix an even number $n \geq 1$ and define
$$M := \max\{S_0, \ldots, S_n\} .$$
   (a) Find the distribution of M.
   (b) Calculate $\mathrm{E}(M - |S_n|)$.

4. A gambler enters a casino with C rupees, where $C \in \mathbb{N}$. At each game, the gambler bets Re. 1, and gets back either Rs. 2 or nothing, each with probability $1/2$. Given that after n games, he has won K rupees in n many games, find the conditional probability that he was out of money at some point in the middle.

5. Another gambler, in the same Casino as above, plans to leave the moment she has won K rupees, where $K \in \mathbb{N}$. Find the probability that she achieves this in n games.

6. Find the probability that a symmetric random walk starting from 0 hits 0 for the first time after $2n$ steps.

7. If n left braces and n right braces are randomly arranged, find the probability that the arrangement is legitimate. An expression is legitimate if at no point, the number of right braces to the left of it exceeds the number of left braces to the left of it.

8. Let $(S_n : n \geq 0)$ be a symmetric random walk. For a fixed $k \geq 1$, define
$$S'_n = S_{k+n} - S_k \,, n \geq 0 \,.$$
Show that for any $n \geq 0$, $(S'_0, \ldots, S'_n)$ is independent of S_k and has the same distribution as $(S_0, \ldots, S_n)$.

9. Let $(S_n : n \geq 0)$ be a possibly asymmetric random walk, that is, it starts from zero, takes a positive step with probability p for some $p \in (0, 1)$, and a negative step with probability $1-p$. For every $n \geq 1$, find the distribution and expectation of S_n.

10. Let (S_n) be the possibly asymmetric random walk defined in the above problem. Fix $n \geq 1$ and define
$$M := \max\{S_0, \ldots, S_n\} \,.$$
For integers j and k with $k > \max\{j, 0\}$, calculate
$$P(M \geq k, S_n = j) \,.$$