Probability theory 1

Homework 7

December 29, 2021

1. Calculate the PGFs of the following distributions.
   (a) Negative Binomial(k, p)
   (b) Binomial(n, p)
   (c) Poisson(λ)
   (d) Geometric(p)
   (e) Uniform on $\{1, \ldots, n\}$

2. If $X_1, \ldots, X_k$ are i.i.d. from Geometric(p), show with the help of PGF that
$$X_1 + \ldots + X_k \text{ follows Negative Binomial}(k, p)\,.$$

3. Let $X_1, \ldots, X_k$ be independent random variables. Suppose that
$$X_i \sim \text{Binomial}(n_i, p),\ 1 \le i \le k\,.$$
Show, with the help of PGF, that
$$\sum_{i=1}^{n} X_i \sim \text{Binomial}\left(\sum_{i=1}^{k} n_i\,,\, p\right)\,.$$

4. Let $X_1, \ldots, X_k$ be independent Poisson random variables with parameters $\lambda_1, \ldots, \lambda_k$ respectively. Find the distribution of $\sum_{i=1}^{k} X_i$.

5. If $X_1, \ldots, X_k$ are i.i.d. such that
$$X_1 + \ldots + X_k \text{ follows Poisson}(\lambda)\,,$$
show that X_1 follows Poisson(λ/k).

6. Let $X_1, X_2, \ldots$ be i.i.d. random variables, taking values in $\mathbb{N} \cup \{0\}$ with PGF ϕ. Suppose that N is a random variable, independent of $(X_1, X_2, \ldots)$, taking non-negative integer values and having PGF ψ. Find the PGF of
$$Z := X_1 + \ldots + X_N\,.$$

7. In the above problem, consider the special case where X_i's follow Bernoulli(p) and N follows Poisson(λ). Find the distribution of Z. Does this remind of any previous exercise?

8. Let X be a random variable taking non-negative integer values and having PGF f. Suppose that $Y \sim$ Bernoulli(p) independently of X. Calculate the PGF of XY.

9. Let X_n follow Binomial(n, p_n) where p_n is such that

$$\lim_{n\to\infty} np_n = \lambda \in (0, \infty) .$$

Show with the help of PGF that

$$\lim_{n\to\infty} P(X_n = k) = P(Y = k) \, , k = 0, 1, 2, \dots \, ,$$

where Y follows Poisson(λ).

10. Let f and g be two PGFs. Show that the following are also PGFs of some random variable taking non-negative integer values:

    (a) fg,

    (b) $pf + (1-p)g$ where $0 < p < 1$,

    (c) $f \circ g$.

11. Let $X_1, X_2, \dots, X_\infty$ be random variables, taking values in $\mathbb{N} \cup \{0\}$, with respective PGFs $f_1, f_2, \dots, f_\infty$. If it holds that

$$\lim_{n\to\infty} P(X_n = k) = P(X_\infty = k) \text{ for every } k = 0, 1, 2, \dots \, ,$$

then show that

$$\lim_{n\to\infty} \sup_{|s|\le 1} |f_n(s) - f_\infty(s)| = 0 .$$

12. Suppose that $X_1, X_2, \dots$ are random variables, taking non-negative integer values, with PGFs $\phi_1, \phi_2, \dots$, respectively. For every $s \in (-1, 1)$,

$$\lim_{n\to\infty} \phi_n(s) = \phi(s) .$$

Show that there exists a random variable Z such that

$$\lim_{n\to\infty} P(X_n = k) = P(Z = k) \text{ for every } k \in \mathbb{N} \cup \{0\} \, ,$$

if and only if

$$\lim_{s\uparrow 1} \phi(s) = 1 .$$

13. Let $X_1, X_2, \ldots$ be i.i.d. random variables taking values $0, 1, 2, \ldots$. For $n \geq 1$, let $Y_1, \ldots, Y_n$ be i.i.d. Bernoulli$(1/n)$ random variables, independent of $(X_1, X_2, \ldots)$. Define

$$Z_n := \sum_{i=1}^{n} X_i Y_i \,.$$

(a) Show that there exists a non-negative integer valued random variable Z such that

$$\lim_{n\to\infty} P(Z_n = k) = P(Z = k) \text{ for each } k = 0, 1, 2, \ldots \,.$$

(b) Calculate $P(Z = 0)$ and $P(Z = 1)$ in terms of p_0, p_1 etc., where $p_k = P(X_1 = k)$.

(c) Convince yourself that showing (a) without using PGFs is difficult.

14. Let the random variables $X_1, X_2, \ldots$ be i.i.d. taking values in the set $\mathbb{N} \cup \{0\}$ and so be $Y_1, Y_2, \ldots$, though their distribution may **possibly** be different from that of X_1.

(a) If it holds that for some $k \geq 2$,

$$X_1 + \ldots + X_k \stackrel{d}{=} Y_1 + \ldots + Y_k \,,$$

then show that

$$X_1 \stackrel{d}{=} Y_1 \,.$$

(b) Suppose that N is a random variable taking values in $\mathbb{N} \cup \{0\}$ with $P(N = 0) < 1$, independent of $(X_1, X_2, \ldots, Y_1, Y_2, \ldots)$. If it holds that

$$X_1 + \ldots + X_N \stackrel{d}{=} Y_1 + \ldots + Y_N \,,$$

then show that

$$X_1 \stackrel{d}{=} Y_1 \,.$$

15. A fair die is rolled till the first six is obtained. Let Z be the sum of the numbers obtained on all the rolls, including the last roll which gives the six. Find the PGF of Z.