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Branching processes (contd.)

Last class: $p_e = \min\{0 \leq s \leq 1 : \phi(s) = s\}$.

As before, $(X_n : n \geq 0)$ is a branching process. The progeny PGF and the probability of extinction are denoted by ϕ and p_e, resp.

Qn 1. At most how many solutions does the following eq^n have:

$$\phi(s) = s, \quad 0 \leq s \leq 1 \quad ?$$

Ans: Unless $p_1 = 1$, at most 2.

Qn 2. When is $p_e = 0$, $p_e = 1$ and $0 < p_e < 1$?

Is the answer related to "R_0" which is the progeny mean?

Qn 3. If $p_e = 1$, that extinction is a.s., what can be said about the extinction time?

If $p_1 = 1$ (the progeny distn puts mass $p_0, p_1, \dots$ on $0, 1, \dots$, resp), then $\phi(s) = s$ for all $0 \le s \le 1$.

<u>Thm.</u> If $p_1 \neq 1$, then

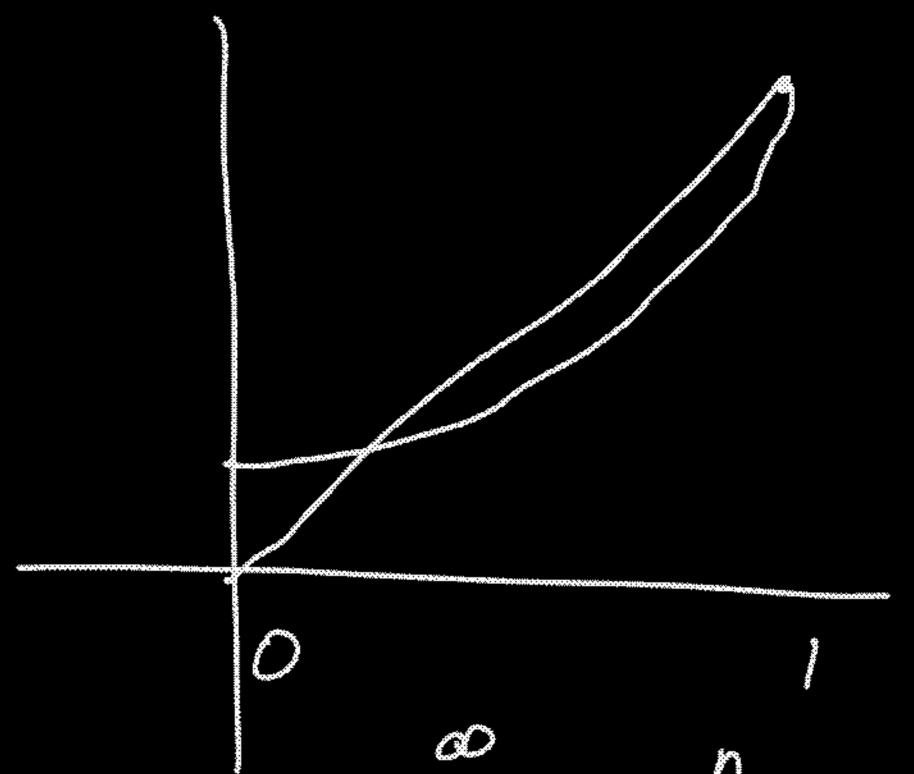
$$\#\{s \in [0,1] : \phi(s) = s\} \le 2 .$$

<u>Proof.</u> Let us first consider the case $p_0 + p_1 = 1$.

In this case,

$$\phi(s) = p_0 + p_1 s .$$

Since $p_1 < 1$, $\{ s \in [0,1] : \phi(s) = s \} = \{ 1 \}$.



Now suppose that $p_0 + p_1 < 1$. Then $p_n > 0$ for some $n \geq 2$. Hence,

$$\phi''(s) = \sum_{n=2}^{\infty} n(n-1) p_n s^{n-2}, \quad 0 \leq s \leq 1,$$

$$\phi(s) = \sum_{n=0}^{\infty} p_n s^n$$

$$\phi'(s) = \sum_{n=1}^{\infty} n p_n s^{n-1}$$

where f' and f'' denote the first and second derivatives, resp, of any function f whenever they exist.

Thus,

$$\phi''(s) > 0 \quad , \quad 0 < s \leq 1 . \qquad (*)$$

If possible, let

$$\phi(s) = s, \quad 0 \leq s \leq 1$$

have 3 solutions, say s_1, s_2, s_3 with $0 \leq s_1 < s_2 < s_3 \leq 1$.

Define $f : [0,1] \to \mathbb{R}$ by

$$f(s) = \phi(s) - s .$$

Then $f(s_1) = f(s_2) = f(s_3)$.

Rolle's theorem implies there exists $\xi_1 \in (s_1, s_2)$ such that

$$f'(\xi_1) = 0 ,$$

and there exists $\xi_2 \in (s_2, s_3)$ with $f'(\xi_2) = 0$.

Apply Rolle once again to get $\xi \in (\xi_1, \xi_2)$ with $f''(\xi) = 0$.

However, since $0 = f''(\xi) = \phi''(\xi)$, this is a contradiction

because $\xi > \xi_1 > s_1 \geq 0$; violating $(*)$.

This completes the proof. $\boxed{/\!/}$

<u>Thm.</u> 1. If $p_0 = 0$, then $p_e = 0$.

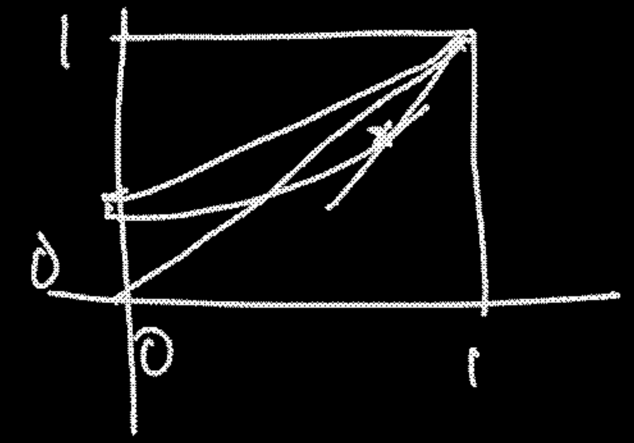
2. If $p_0 > 0$ and $\sum_{n=1}^{\infty} n p_n \leq 1$, then $p_e = 1$.

3. If $p_0 > 0$ and $\sum_{n=1}^{\infty} n p_n > 1$, then $0 < p_e < 1$.

Proof. 1. If $p_0 = 0$, then $\phi(0) = 0$ and hence $p_e = 0$.

The other way of looking at it is $p_0 = 0$ means that each individual gives rise to at least one offspring, in which case, the population cannot die out.

2. Assume $p_0 > 0$ and $\sum_{n=1}^{\infty} n p_n \leq 1$.



If possible, let $p_e < 1$. The mean value theorem implies that for some $\xi \in (p_e, 1)$,

$$\frac{\phi(1) - \phi(p_e)}{1 - p_e} = \phi'(\xi),$$

that is $\phi'(\xi) = 1$ because LHS $= 1$.

Since $p_0 > 0$ and $p_e < 1$, it is not possible that $p_1 = 1 - p_0$ because then $p_e = 1$. Thus $p_1 < 1 - p_0$. However then

$p_n > 0$ for some $n \geq 2$ and hence

$$\phi''(s) > 0 \,, \quad 0 < s \leq 1 \,,$$

as shown in the previous proof.

Therefore $\quad \phi'(1) > \phi'(\xi) = 1 \,,$

but $\quad \phi'(1) = \sum_{n=1}^{\infty} n p_n \leq 1 \,.$

This is a contradiction. Thus, $p_e = 1$ in this case.

3. Assume $p_0 > 0$ and $\sum_{n=1}^{\infty} n p_n > 1$.

Since $\sum_{n=1}^{\infty} n p_n > 1$, there exists N s.t.

$$\sum_{n=1}^{N} n p_n > 1 .$$

Since $\lim_{s \uparrow 1} \sum_{n=1}^{N} n p_n s^{n-1} = \sum_{n=1}^{N} n p_n > 1$, there exists $0 \le s_1 < 1$ such that

$$\sum_{n=1}^{N} n p_n s_1^{n-1} > 1 .$$

Thus,

$$\phi'(s_1) = \sum_{n=1}^{\infty} n p_n s_1^{n-1} \geq \sum_{n=1}^{N} n p_n s_1^{n-1} > 1 .$$

Our next claim is that $\phi(s_1) \le s_1$. If not, then

$$
\begin{aligned}
\phi(1) &= \phi(1) - \phi(\delta_1) + \phi(\delta_1) \\
\text{(we assumed } \phi(\delta_1) > \delta_1 \text{ for the sake of cont)} \quad &> \phi(1) - \phi(\delta_1) + \delta_1 \\
\text{(for some } \xi \in (\delta_1, 1)\text{, by MVT)} \quad &= (1 - \delta_1)\,\phi'(\xi) + \delta_1 \\
(\phi' \text{ is non-dec}) \quad &\geq (1 - \delta_1)\,\phi'(\delta_1) + \delta_1 \\
(\phi'(\delta_1) > 1) \quad &> 1 - \delta_1 + \delta_1 \\
&= \underline{1}\,.
\end{aligned}
$$

Since this is a contradiction, we have $\phi(s_1) \leq s_1$.

Thus, the function $\phi(s) - s$ is strictly positive at 0 because $p_0 > 0$ and non-positive at s_1. Thus

$$\{0 < s \leq s_1 : \phi(s) = s\} \neq \emptyset.$$

Hence $0 < p_e < 1$ as desired. $\square$

	$p_0 = 0$	$p_0 > 0$
$\sum_{n=1}^{\infty} n p_n < 1$	Impossible	$p_e = 1$
$\sum_{n=1}^{\infty} n p_n = 1$	(possible only if $p_1 = 1$) $p_e = 0$	$p_e = 1$
$\sum_{n=1}^{\infty} n p_n > 1$	$p_e = 0$	$0 < p_e < 1$.

This table answers Qn 2.

Exc. Calculate the probability of extinction of a branching process with progeny distribution Bernoulli (p) for $0 \leq p \leq 1$.

Soln. The progeny PGF in this case is

$$\phi(s) = 1 - p + ps .$$

Solving for $\phi(s) = s$, if $p \neq 1$, the only solution is $s = 1$. If $p = 1$, then $\phi(s) = s$ for all $s \in [0,1]$. Thus,

$$p_e = \begin{cases} 1, & 0 \le p < \frac{1}{2}, \\ 0, & p = 1, \end{cases}$$

is the extinction probability of the process.

<u>Defn.</u> Define for a branching process $(X_n : n \geq 0)$, its survival time S by

$$S = \sup\{n \geq 0 : X_n \neq 0\}.$$

The possibility $S = \infty$ is allowed.

Thm. # Assume that

$$m = \sum_{n=1}^{\infty} n p_n < 1 .$$

Then, $$E(S) \leq \frac{m}{1-m} .$$

Exc. If $(X_n : n \geq 1)$ is a branching process with mean progeny finite, denoted by m, then show that

$$E(X_n) = m^n .$$

Soln. Recall that the PGF of X_n is $\phi \circ \cdots \circ \phi$ (n times).

The chain rule implies that

$$\frac{d}{ds}\phi^{(n)}(s) = \phi'\left(\phi^{(n-1)}(s)\right)\phi'\left(\phi^{(n-2)}(s)\right)\cdots\phi'(s).$$

Putting $s=1$, we get using the fact that $\phi^{(n-1)}, \phi^{(n-2)}, \ldots$ are PGFs,

$$\left.\frac{d}{ds}\phi^{(n)}(s)\right|_{s=1} = \phi'(1)\,\phi'(1)\cdots\phi'(1)$$
$$= m^n$$

because the given hypothesis is same as $\phi'(1) = m$. Thus $E(X_n) = m^n$.

Exc. Try showing the above by induction on n.

Exc. If Z is a random variable taking values $0, 1, 2, \ldots$, then

$$P(Z \geq 1) \leq E(Z).$$

Soln. Follows by recalling

$$E(Z) = \sum_{n=1}^{\infty} P(Z \geq n).$$

$$\begin{aligned} E(Z) &= 1\,P(Z=1) \\ &\quad + 2\,P(Z=2) + \cdots \\ &= P(Z=1) \\ &\quad + P(Z=2) + P(Z=2) + \\ &\quad + P(Z=3) + P(Z=3) + P(Z=3) \\ &\quad + \cdots \\ &= P(Z \geq 1) + P(Z \geq 2) + P(Z \geq 3) + \cdots \end{aligned}$$

Proof of Theorem #. The assumption implies that $S < \infty$ a.s. Thus, S takes values in $\mathbb{N} \cup \{0\}$.

Therefore,

$$E(S) = \sum_{n=1}^{\infty} P(S \geq n)$$

$$= \sum_{n=1}^{\infty} P(X_n \geq 1)$$

$$\left(\text{the immediately above exc}\right) \leq \sum_{n=1}^{\infty} E(X_n)$$

$$\left(\text{the exc before the above}\right) = \sum_{n=1}^{\infty} m^n$$

$$= \frac{m}{1-m} . \qquad \blacksquare$$

Exc. Consider a branching process with progeny distn

$(p_n : n \geq 0)$ where $p_0 = p_1 = p_2 = p_3 = \frac{1}{4}$ and $p_4 = p_5 = \dots = 0$.

Calculate its extinction probability.

Soln. The progeny PGF is

$$\phi(s) = \frac{1}{4}\left(1+s+s^2+s^3\right).$$

To solve $\phi(s) = s$, write

$$4\left[\phi(s) - s\right] = s^3 + s^2 - 3s + 1$$

$$= s^2(s-1) + 2s(s-1) - (s-1)$$

$$= (s-1)(s^2 + 2s - 1)$$

$$= (s-1)\left(s - \frac{-2+\sqrt{4+4}}{2}\right)\left(s - \frac{-2-\sqrt{4+4}}{2}\right).$$

$$\frac{-b \pm \sqrt{b^2-4ac}}{2a}.$$

The solutions of $\phi(s) = s$, $s \in \mathbb{R}$ are 1, $\sqrt{2}-1$, $-\sqrt{2}-1$.

Thus, $\{ s \in [0,1] : \phi(s) = s \} = \{ \sqrt{2} - 1 , 1 \}$.

Hence the probability of extinction is $\sqrt{2} - 1$.

<u>Exc.</u> Consider a population starting from a single cell and evolving as follows. The cell tosses a coin with probability of heads p, for some $0 < p < 1$. If a tail is obtained, the cell gives birth to another cell. If a head is obtained, the cell dies.

This is continued until the first head is obtained. Each cell in the population follows the same procedure independently. Calculate the probability that eventually there will be no alive cells.

<u>Soln.</u> This is a branching process with progeny distn $(p_n : n \geq 0)$ where $p_n = (1-p)^n p$, $n \in \{0,1,2,\ldots\}$.

The progeny PGF is

$$\phi(s) = \sum_{n=0}^{\infty} p_n s^n$$

$$= \sum_{n=0}^{\infty} (1-p)^n p s^n$$

$$= \frac{p}{1-(1-p)s}.$$

Solving for $\phi(s) = s$ is same as

$$p = s - (1-p)s^2,$$

that is, $(1-p)s^2 - s + p = 0$,

whose roots are $\dfrac{1 \pm \sqrt{1-4p(1-p)}}{2(1-p)}$.

Write

$$1 - 4p(1-p) = \left(p + (1-p)\right)^2 - 4p(1-p)$$

$$= p^2 + (1-p)^2 - 2p(1-p)$$

$$= \left(p - (1-p)\right)^2 = (2p-1)^2.$$

Thus, $\dfrac{1 \pm |2p-1|}{2(1-p)}$ are the roots, that is, $\dfrac{1+(2p-1)}{2(1-p)}$ and $\dfrac{1-(2p-1)}{2(1-p)}$,

that is, $\dfrac{p}{1-p}$ and 1.

The prob of extinction is

$$p_e = \begin{cases} \dfrac{p}{1-p}, & 0 < p < \dfrac{1}{2}, \\ 1, & \dfrac{1}{2} \le p < 1. \end{cases}$$

Exc. Calculate the extinction probability of a branching process with progeny distn $\mathrm{Bin}(n, p)$ for $n = 1, 2, 3$ and $0 \le p \le 1$.

$\boxed{\text{Karlin \& Taylor}}$