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Probability theory 1

Weightage:

Mid-term exam : 40%

Final exam : 60%

Tuesday : 2 - 3:55
Wednesday : 11:30 - 12:25
Friday : 2 - 3:55

Feller Vol. 1.

Chances of rain tomorrow are 70%.

What does this mean?

If several days similar to tomorrow are considered, proportion of rainy days among them is close to 0.7.

One goal: Making perfect mathematical sense of the above sentence.

Random experiment:

A random experiment is an experiment whose set of possible outcomes is known in advance but the actual outcome is unknown.

Examples: Toss a coin, Roll a die, Choosing a card at random from a deck.

Defn. A sample space is the set of all possible outcomes of a random experiment. It is customary to denote a sample space by Ω.

Example 1: Tossing a coin: $\Omega = \{\text{Head}, \text{Tail}\}$.

Example 2: Roll a die: $\Omega = \{1, 2, 3, 4, 5, 6\}$.

Example 3: Pick a card from a standard deck.

$$\Omega = \Big\{ (r, s) : r \in \{\text{Ace}, 2, 3, \ldots, 10, \text{Jack}, \text{Queen}, \text{King}\}$$
$$s \in \{\text{Clubs}, \text{Hearts}, \text{Diamonds}, \text{Spades}\} \Big\}.$$

<u>Example 4</u>: Toss a coin 5 times. Then the sample space is

$$\Omega = \Big\{ (i_1, i_2, i_3, i_4, i_5) : i_j \in \{H, T\} \text{ for } j = 1, \ldots, 5 \Big\}.$$

Defn. Elements of the sample space are "outcomes". Subsets of the sample space are "events".

Example: Roll a fair die 5 times. Let A be the event that the sum of the numbers obtained is even.

The sample space is

$$\underline{\Omega} = \{(i_1, \ldots, i_5) : i_j \in \{1, 2, \ldots, 6\} \text{ for } j = 1, \ldots, 5\}.$$

$$A = \left\{ (i_1, \dots, i_5) \in \Omega : \sum_{j=1}^{5} i_j \text{ is an even number} \right\}.$$

<u>Example</u>: Toss a coin twice and let E be the event that the outcomes of the 2 tosses are different. The sample space is

$$\Omega = \{HH, HT, TH, TT\},$$

$$E = \{HT, TH\}.$$

Classical definition of probability

Consider a random experiment whose sample space Ω has a finite cardinality and all outcomes of the experiment are equally likely. Then, for any event A, its probability $P(A)$ is defined by

$$P(A) = \frac{\# A}{\# \Omega},$$ where $\# E$ denotes the cardinality of the set E.

Example: A fair coin is tossed 2 times. Calculate the probability that the outcomes of the 2 tosses are distinct.

For this example, the sample space is

$$\Omega = \{HH, HT, TH, TT\}$$

and the event in the question is

$$E = \{HT, TH\}.$$

Using the classical definition of probability, we get

$$P(E) = \frac{\# E}{\# \Omega} = \frac{2}{4} = \frac{1}{2}.$$

Example: A fair die is rolled 7 times. Calculate the probability that the sum of the numbers obtained is odd.

The sample space is

$$\Omega = \{ (i_1, \dots, i_7) : i_j \in \{1, \dots, 6\} \text{ for } j = 1, \dots, 7 \}.$$

Therefore, $\# \Omega = 6^7$.

The event in the question is

$$\mathcal{O} = \left\{(i_1,\dots,i_7)\in\Omega : \sum_{j=1}^{7} i_j \text{ is an odd number}\right\}.$$

For calculating $\#\mathcal{O}$, we write

$$\#\mathcal{O} = \#\left\{(i_1,\dots,i_7)\in\Omega : \sum_{j=1}^{6} i_j \text{ is odd and } \sum_{j=1}^{7} i_j \text{ is odd}\right\}$$

$$+\ \#\left\{(i_1,\dots,i_7)\in\Omega : \sum_{j=1}^{6} i_j \text{ is even and } \sum_{j=1}^{7} \text{ is odd}\right\}$$

$$= \#\left\{(i_1,\dots,i_7)\in\Omega : \sum_{j=1}^{6} i_j \text{ is odd and } i_7\in\{2,4,6\}\right\}$$

$$+ \# \left\{ (i_1, \dots, i_7) \in \Omega : \sum_{j=1}^{6} i_j \text{ is even and } i_7 \in \{1, 3, 5\} \right\}$$

$$= 3 \# \left\{ (i_1, \dots, i_6) : i_j \in \{1, \dots, 6\} \text{ and } \sum_{j=1}^{6} i_j \text{ is odd} \right\}$$

$$+ 3 \# \left\{ (i_1, \dots, i_6) : i_j \in \{1, \dots, 6\} \text{ '' '' '' even} \right\}$$

$$= 3 \# \Bigg[\left\{ (i_1 \dots i_6) : 1 \leq i_j \leq 6 \text{ and } \sum_{j=1}^{6} i_j \text{ is odd} \right\} \cup$$

$$\left\{ (i_1, \dots, i_6) : 1 \leq i_j \leq 6 \text{ and } \sum_{j=1}^{6} i_j \text{ is even} \right\} \Bigg]$$

[because the 2 sets are disjoint, that is, their intersection is the empty set]

$= 3 \, \# \left\{ (i_1, \dots, i_6) : i_j \in \{1, \dots, 6\} \text{ for } j = 1, \dots, 6 \right\}$

$= 3 \cdot 6^6.$

Thus,

$$P(O) = \frac{3 \cdot 6^6}{6^7} = \frac{3}{6} = \frac{1}{2}.$$

Example. A fair coin is tossed 10 times. What is the probability of getting 7 Heads?

Soln. In this example, the sample space is

$$\Omega = \{(i_1, \ldots, i_{10}) : i_j \in \{H, T\} \text{ for } j = 1, \ldots, 10\}.$$

Thus, $\# \Omega = 2^{10}$.

Denote

A = Event that we get 7 Heads

$= \{(i_1, \dots, i_{10}) \in \Omega : i_j = H \text{ for exactly } 7 \text{ } j\text{'s}\}.$

Thus $\# A = \binom{10}{7} = \frac{10!}{7!\,3!}.$

Hence, $P(A) = \frac{\# A}{\# \Omega} = 2^{-10} \binom{10}{7}.$

Example: In the above example calculate the probability that an even number of Heads is obtained.

Soln. Let E be the event that the number of Heads obtained is even. Then

$$\# E = \#[\text{Getting } 0 \text{ H's}] + \#[\text{Getting } 2 \text{ H's}] + \cdots + [\text{Getting } 10 \text{ H's}]$$

$$= \binom{10}{0} + \binom{10}{2} + \binom{10}{4} + \cdots + \binom{10}{10}.$$

Recalling the binomial theorem

$$\binom{10}{0}+\binom{10}{1}+\binom{10}{2}+\cdots+\binom{10}{10}=(1+1)^{10}=2^{10}$$

$$\binom{10}{0}-\binom{10}{1}+\binom{10}{2}-\cdots+\binom{10}{10}=(1-1)^{10}=0$$

$$2\left[\binom{10}{0}+\binom{10}{2}\cdots+\binom{10}{10}\right]=2^{10}$$

Thus, $\binom{10}{0}+\binom{10}{2}+\binom{10}{4}+\cdots+\binom{10}{10}=2^{9}$.

That is, $\# E=2^{9}$. Hence $P(E)=\dfrac{2^{9}}{2^{10}}=\dfrac{1}{2}$.

Set-theoretic notions

For sets A, B,

$A \cup B = \{\omega : \omega \in A$ or $\omega \in B\}$, (union of A and B)

$A \cap B = \{\omega : \omega \in A$ and $\omega \in B\}$. (intersection of A and B).

$A \setminus B = \{\omega : \omega \in A$ and $\omega \notin B\}$ (A setminus B)

$A^c = \{\omega : \omega \notin A\}$.

$$A \triangle B = (A \setminus B) \cup (B \setminus A)$$ (symmetric difference of A and B).

Defn. For events $A_1, \dots, A_n$, their union is defined as

$$\bigcup_{i=1}^{n} A_i = \{\omega : \omega \in A_i \text{ for } \underline{\text{some}}\ i \text{ in } \{1, \dots, n\}\},$$

their intersection is defined as

$$\bigcap_{i=1}^{n} A_i = \{\omega : \omega \in A_i \text{ for } \underline{\text{every}}\ i \text{ in } \{1, \dots, n\}\}.$$

The events $A_1, \ldots, A_n$ are "mutually exclusive" if for all $1 \le i \ne j \le n$, A_i and A_j are disjoint, that is, $A_i \cap A_j = \phi$.

null set or the empty set.

If $\bigcup_{i=1}^{n} A_i = \Omega$, where Ω is the sample space, then $A_1, \ldots, A_n$ are "exhaustive".

<u>Fact</u>: If $A_1, \dots, A_n$ are mutually exclusive finite sets,

then

$$\# \bigcup_{i=1}^{n} A_i = \sum_{i=1}^{n} \# A_i .$$