Probability theory 1

22/9/21

We have known about:

Random experiment, Sample space, Sample points, Event.

Some examples are shown.

Classical definition of probability

Consider a random experiment whose sample space Ω has a finite cardinality and all outcomes of the experiment are equally likely. Then for an event A (that is $A \subseteq \Omega$), its probability is defined by

$$P(A) = \frac{\# A}{\# \Omega}$$

where $\# E$ denotes the cardinality of the set E.

Proceed with some more examples.

- <u>Example</u>: Let there be 8 distinct balls and they are put into 8 distinct boxes randomly. What is the probability that each box will be occupied?
- The sample space Ω has cardinality 8^8.

The event A given in the question may be performed in

$$8 \times 7 \times \cdots \times 1 = 8! \text{ ways.}$$

So, using the classical definition of probability we get,

$$P(A) = \frac{\# A}{\# \Omega} = \frac{8!}{8^8}.$$

- In general, placing n distinct balls into n distinct boxes such that each box is occupied has probability $\dfrac{n!}{n^n}$.

Example: Two cards are chosen at random from a deck (standard) of cards without replacement. What is the probability that the two cards either have the same suite or the same rank?

- The cardinality of the sample space Ω, is

$$\binom{52}{2}$$

Suppose $\# E_s :=$ number of ways to get 2 same suite cards

$$= 4\binom{13}{2}$$

$\# E_r :=$ number of ways to get 2 same rank cards

$$= 13 \binom{4}{2}$$

It is clear that it is impossible to get 2 same rank and same suits cards.

Therefore, our requied event E has cardinality

$$\# E_s + \# E_r .$$

Using the classical definition of probability we get,

$$P(E) = \frac{\# E}{\# \Omega} = \frac{4\binom{13}{2} + 13\binom{4}{2}}{\binom{52}{2}} = \frac{5}{17} .$$

<u>Example</u>: A fair die is rolled 5 times. Find the probability that the sum of the numbers observed is 15.

· The number of possible ways in which 15 is the sum of the numbers observed in 5 rolls of the die equals the coefficient of x^{15} in

$$f(x) = (x + x^2 + x^3 + \cdots + x^6)^5 \quad , \quad |x| < 1$$

$$2 + 4 + 3 + 3 + 3 = 15$$

$$(x + x^{2} + \cdots + x^6)(x + x^2 + x^{?} + x^6) \cdots \cdots (x + x^2 + x^{3} \cdots + x^6) \sim x^{15}$$

(The number of elements in the sample space $=: \underline{\#\Omega = 6^5}$)

※ ~~Coeffient~~ Coefficient of x^{15} in the polynomial $x^5(1+x+x^2+\cdots+x^5)^5$

$$= x^5\left\{\frac{(1-x^6)}{(1-x)}\right\}^5 \quad \left[\begin{array}{l}(1-x)(1+x+\cdots+x^5)\\ = 1-x^6\\ x \neq 1\end{array}\right]$$

$\parallel$

$$= x^5(1-x)^{-5}(1-x^6)^5$$

Coefficient of x^{10} in the polynomial $(1-x)^{-5}(1-x^6)^5$, $|x|<1$

$$= \left(1+5x+\frac{5(5+1)}{2!}x^2+\frac{5(5+1)(5+2)}{3!}x^3+\frac{5(5+1)(5+2)(5+3)}{4!}x^4+\cdots\right)\times$$

$\cancel{x^{10}}$

$$\binom{5+3}{4}\ ''$$

$$\left(\underbrace{1}_{4} - 5x^{6} \pm \cdots\cdots \right)$$

$$= \binom{5+9}{10} - 5\binom{5+3}{4} = 651 = \#E$$

Using the classical definition of probability.

$$P(E) = \frac{\#E}{\#\Omega} = \frac{651}{6^5}$$