24/9/21

Probability theory 1

Feller - Probability theory - Vol 1.

Sheldon and Ross - A first course in Probability

We have discussed the classical definition of probability and solved some examples.

Let us proceed with the following example:

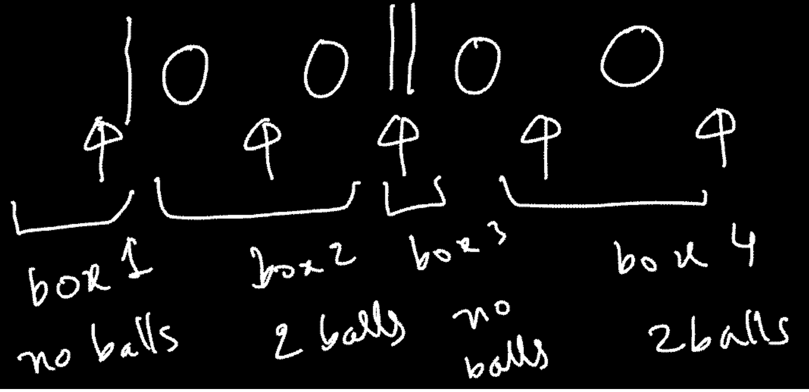
Bose-Einstein experiment:

Suppose we have n many identical balls. They are put into k many distinct boxes. We want to calculate the possible number of ways to perform this act.

Method 1:

Rough!

4 identical balls 4 distinct boxes



$$\underbrace{0}_{\text{box 1}} \mid \underbrace{0}_{\text{box 2}} \mid \underbrace{0 \quad 0}_{\text{box 3}} \mid \underbrace{\;}_{\text{box 4}}$$

box 1 — 1 ball

box 2 — 1 ball

box 3 — 2 balls

box 4 — no balls

111 222

1 2 1 2 1 2

1 1 2 2 1 2

$$\frac{6!}{3!\,3!} = \binom{6}{3}$$

0000111

$$\frac{7!}{4!\,3!} = \binom{7}{3}$$

$$\underbrace{0|0||\cdots|0}_{4}$$

n balls (identical) and n boxes (distinct)

$$\Big\{\underbrace{0 \cdots 0}_{n \text{ identical balls}}\Big\} \Big\{\underbrace{11 \cdots 1}_{k-1}\Big\} \qquad \frac{(n+k-1)!}{n!\,(k-1)!} = \binom{n+k-1}{k-1}$$

Method 2: The required number of ways is same as the cardinality of the following set

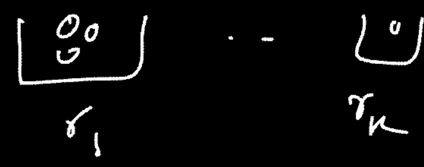
$$\left\{ (r_1, r_2, \cdots, r_k) : r_i \in \mathbb{N} \cup \{0\},\ \sum_{i=1}^{k} r_i = n \right\}$$

#balls in box 1

#balls in box 2

#balls in box k

$\mathbb{N} = \{1, 2, 3, \cdots\}$



The cardinality of the above set is same as the co-efficient of x^n in the expansion of $f(x) = (1 + x + \cdots + x^n)^k$, $|x| < 1$

$$(1+x+\cdots+x^n)(1+x+\cdots+x^n)\cdots\cdots(1+x+\cdots+x^n)$$

$$\downarrow \qquad\qquad \downarrow \qquad\qquad \downarrow$$

$$x^{r_1} \qquad x^{r_2} \qquad x^{r_k} = x^{r_1+r_2+\cdots+r_k}$$

$$= x^n$$

= The coefficient of x^n in $(1+x+x^2+\cdots)^k$, $|x|<1$

$$= (1-x)^{-k}$$

The coefficient of x^n in

$$= 1+kx+\frac{k(k+1)}{2!}x^2+\cdots+\underbrace{\frac{k(k+1)\cdots(k+n-1)}{n!}}x^n+\cdots$$

$$= \binom{k+n-1}{n} = \binom{k+n-1}{k-1} \quad \square$$

<u>Example</u>: If n identical balls are thrown into k distinct boxes, with $n \geq k$, find the probability that all the boxes have at least one ball.

. The number of elements in the sample space is $\binom{n+k-1}{k-1}$.

what is the number of ways, that we can put at least 1 ball in each boxes?

We calculate the coefficient of x^n in $f(x) = (x + x^2 + \cdots + x^n + \cdots)^k$

$$(x + x^2 + \overset{x^{r_1}}{\cdots})(x + x^2 + \overset{x^{r_2}}{\cdots}) \cdots (x + x^2 + \overset{x^{r_k}}{\cdots}) \quad k \text{ terms}$$

$\downarrow$ box 1, r_1 balls

$\downarrow$ box 2, r_2 balls

$\downarrow$ box k, r_k alls

$x^{r_1} \cdot x^{r_2} \cdots x^{r_k} = x^{r_1 + \cdots + r_k}$

the coefficient of x^n in $x^k(1+x+\cdots+\cdots)^k$

$=$ the coefficient of x^{n-k} in $(1+x+\cdots)^k$, $|x|<1$

$$= \binom{n-k+k-1}{k-1} = \binom{n-1}{k-1}$$

coefficient of x^n in $(1+x+\cdots)^k$ is $\binom{n+k-1}{k-1}$

$n \to n-k$

Using the classical definition of probability we get that the required probability is

$$\frac{\binom{n-1}{k-1}}{\binom{n+k-1}{k-1}}$$

Limitations of the definition of classical probability:

Consider the following experiments

1) Toss a coin with chances of Heads is 0.3.
   (the outcomes are not equally likely)
2) Tossing a fair coin untill we get the first tail.
   (the sample space is infinite)

Recall: Let A and B be two sets. We call A and B to be bijective if A and B are injective and surjective.

- A and B are injective if there exist a function $f: A \to B$ such that for any ($x, y \in A$ if $f(x) = f(y) \Rightarrow x = y$)

(in other words for any $x, y \in A$, $x \neq y \Rightarrow f(x) \neq f(y)$).

such a function f is called an injective function or injection.

- A and B are surjective if there exist a function $g: A \to B$ such that for any $b \in B$ there exists a $x \in A$ such that

$$g(x) = b.$$

- Fact; $f: A \to B$ is an injection $\Rightarrow$ $|A| \leq |B|$ (or $\# A \leq \# B$)

$g : A \to B$ is a surjection $\Rightarrow$ $\underline{|A| \geq |B|}$ (or $\# A \geq \# B$)

If $\underline{A \text{ and } B \text{ are bijective}}$ then $|A| = |B|$ ($\# A = \# B$)

$\underline{\text{pf:}}$ $|A| \leq |B| \leq |A| \Rightarrow |A| = |B|$.

- $\mathbb{N} = \{1, 2, \cdots\} \to$ define that the cardinality of $\mathbb{N} = |\mathbb{N}| = \# \mathbb{N}$ to be countably infinite.

$\underline{\text{Definition:}}$ $\underline{\text{(finite set)}}$

A non empty set X is called finite if there exist a $\underline{n \in \mathbb{N}}$ such that X and the set $\{1, 2, \cdots, n\}$ is bijective.

It is defined that the null set (denoted by ϕ) has cardinality 0.

<u>Example:</u> $\{a, b, c\}$ is finite ?

$h : \{1, 2, 3\} \longrightarrow \{a, b, c\}$ defined by $h(1) = a$, $h(2) = b$, $h(3) = c$.

We can clearly see that h is a bijection.

Form the definition of finite sets we may conclude that the set $\{a, b, c\}$ is finite.

<u>Definition</u> (<u>Countably infinite sets</u>)

A set X is said to be countably infinite if there exist a bijection between X and $\mathbb{N}$.

<u>Example:</u> $\{a_1, a_2, a_3, \cdots, a_n, \cdots\}$ is countably infinite?

$h : \mathbb{N} \to \{a_1, a_2, \cdots\}$ defined by $h(n) = a_n \quad \forall\, n \in \mathbb{N}$

is a bijection.

$A = \{1, 2\}$

$B = \{1, 2, 3\}$

<u>Definition:</u> (<u>Countable set</u>)

A set X is called countable if it is either finite or countably infinite.

- We shall have our sample space at most a countable set.