Probability theory 1

28/9/21

- Sheldon M. Ross – A first course in probability.
- A home work sheet will be given to you soon.
- The course website is

  www.isical.ac.in/~arijitc/PT1/

- The black board remains black in background.
- For two sets A and B. We call

A is injective to/into B, (A ~~and~~ B ~~injective~~ or ~~surjective~~)
A is surjective to/onto B,

A and B are bijective.

Limitations of the classical definition of probability :

Our sample space will be countable.
In order to accommodate these experiments, we expand
the definition of a probability space as follows:

Definition: Let $\Omega \subseteq \{\omega_1, \omega_2, \dots\}$ be a finite or countably infinite set. A probability assignment is a function

$$p : \Omega \to [0,1]$$

such that $\sum_{i:\omega_i \in \Omega} p(\omega_i) = 1$. For every event $A \, (\subseteq \Omega)$ the probability of A is defined as

$$P(A) = \sum_{i:\omega_i \in A} p(\omega_i).$$

The tuple (Ω, P) is called a probability space.

Example: A coin with chances of Heads p_0 is tossed 2 times independenty. $\left(0 \leq p_0 \leq 1,\ p_0 \text{ is a real number } p_0 \in \mathbb{R} \right)$

(the outcomes may not be equally likely)

- Here the sample space Ω is given by

$$\Omega = \{ (H,T), (T,H), (H,H), (T,T) \}$$

and the probability assignment function $p : \Omega \to [0,1]$ is given by

$$p(\underset{\omega_1}{\underset{\shortparallel}{H,T}}) = p_0(1-p_0) \qquad p(\underset{\omega_2}{\underset{\shortparallel}{T,H}}) = (1-p_0)p_0$$

$$p(\underset{\omega_3}{\underset{\shortparallel}{H,H}}) = p_0^2 \qquad p(\underset{\omega_4}{\underset{\shortparallel}{T,T}}) = (1-p_0)^2.$$

Observe that $\sum_{i:\omega_i \in \Omega} p(\omega_i) = 1$.

<u>Example</u>: A coin with chances of Heads p $(0 \le p < 1)$ is tossed till the first Tails appears. (the sample space is infinite)

Here the sample space Ω is given by

$$\Omega = \{T, HT, HHT, \ldots, \underbrace{HH\cdots H}_{n \text{ times}}T, \ldots\}$$

Let $\omega_n = \underbrace{H \cdots H}_{n-1 \text{ times}} T$, $n \geq 1$. $\omega_1 = T$, $\omega_4 = HHHT$

✓ $p(\omega_1) = p(T) = 1-p$

✓ $p(\omega_2) = p(HT) = p(1-p)$

✓ $p(\omega_n) = p^{n-1}(1-p)$ . . .

Now $\displaystyle\sum_{n:\omega_n \in \Omega} p(\omega_n) = \sum_{n:\omega_n \in \Omega} p^{n-1}(1-p) = \sum_{n=1}^{\infty} p^{n-1}(1-p)$

$$= (1-p)\sum_{n=1}^{\infty} p^{n-1} = (1-p)\frac{1}{1-p} = 1.$$

In the above example what is the probability that we need at least 7 tosses to get the first Tails?

- The required event A is given by

$$A = \{ \underset{\omega_7}{\underset{\parallel}{HHHHHHT}}, \underbrace{HH\cdots H}_{7 \text{ HeAds}}T = \omega_8, \ldots \}$$

The probability of the above event is

$$P(A) = \sum_{i:\omega_i \in A} p(\omega_i) = \sum_{i=7}^{\infty} p^{i-1}(1-p) = (1-p)p^5 \sum_{i=1}^{\infty} p^i$$

$$= (1-p)\,p^5 \,\frac{p}{(1-p)} = p^6 .$$

$$p(\omega_7) \quad , \quad \mathbb{P}(\{\omega_7\})$$

$$A = \{\omega_7\} \subseteq \Omega$$

<u>Example</u>: Toss a coin with probability the chances of Heads is p and toss that 10 times. Calculate the probability that exactly 4 Heads are obtained.

· Here the sample space Ω is given by

$$\Omega = \{ (a_1, a_2, \cdots, a_{10}) : a_i \in \{H, T\} \ \forall \ i = 1, 2, \cdots, 10 \}$$

$$\boxed{p((a_1, a_2, \cdots, a_{10})) := p^k (1-p)^{10-k}}$$ if there is k many Heads and $10-k$ many Tails.

$p((a_1, a_2, \cdots, a_{10})) = \mathbb{P}(\{(a_1, \cdots, a_{10})\})$

$(H, T, T, H, \cdots, H)$ k many Heads & $10-k$ many tails.

$(a_1, \cdots, a_{10}) = \omega_1$

$$p(a_1, a_2) = p(1-p)$$

$$p(H, T) \qquad p(T, H)$$

We need to check $\displaystyle\sum_{(a_1, \cdots, a_{10}) \in \Omega} p((a_1, \cdots, a_{10})) = 1$

$$\sum_{(a_1,\cdots,a_{10})\in\Omega} p\left((a_1,\cdots,a_{10})\right)$$

$$= \sum_{(a_1,\cdots,a_{10})=(H,H,\cdots,H)} p\left((a_1,\cdots,a_{10})\right) + \sum_{\substack{(a_1,\cdots,a_{10})\in\Omega \\ :\exists\ 9\text{ Heads \& } \\ 1\text{ Tail}}} p\left((a_1,\cdots,a_{10})\right) + \sum_{\substack{(a_1,\cdots,a_{10})\in\Omega \\ :\exists\ 8\text{ Heads} \\ \&\ 2\text{ Tails}}} + \cdots$$

$$= 1\times p^{10} + \binom{10}{9} p^9(1-p) + \binom{10}{8} p^8(1-p)^2 + \cdots$$

$$= (p+1-p)^{10} = 1$$

Our required event has probability $P(A)$ where $A = \{(a_1, a_2, \ldots, a_{10}) :$ there are exactly 4 a_i's as H's and the rest are T's $\}$

So, $$P(A) = \sum_{(a_1, \ldots, a_{10}) \in A} p((a_1, \ldots, a_{10})) = \binom{10}{4} p^4 (1-p)^6 .$$