Probability theory 1

29/9/21

We have seen the definition of probability space and discussed some examples.

Theorem 1: (The laws of total probability)

(1) $\mathbb{P}(\phi) = 0$ and $\mathbb{P}(\Omega) = 1$.

Pf: $\mathbb{P}(\phi) = \sum_{\omega \in \phi} p(\omega) = 0$.

$\mathbb{P}(\Omega) = \sum_{\omega \in \Omega} p(\omega) = 1$ (by definition).

(2) For any event E, $\quad 0 \le \mathbb{P}(E) \le 1$.

<u>Lemma:</u> If $A, B \subseteq \Omega$ such that $A \subseteq B$ then $\mathbb{P}(A) \le \mathbb{P}(B)$.

<u>pf of lemma:</u> $\mathbb{P}(A) = \sum_{\omega \in A} p(\omega) \le \sum_{\omega \in B} p(\omega) = \mathbb{P}(B)$.

We observe that $\phi \subseteq E \subseteq \Omega$ and applying the lemma we get $0 = \mathbb{P}(\phi) \le \mathbb{P}(E) \le \mathbb{P}(\Omega) = 1$.

(3) For mutually exclusive events $A_1, A_2, \ldots, A_n$,

$$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \mathbb{P}(A_i).$$

Pf: $\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\omega \in \bigcup_{i=1}^{n} A_i} p(\omega) = \sum_{\omega \in A_1} p(\omega) + \sum_{\omega \in A_2} p(\omega) + \cdots + \sum_{\omega \in A_n} p(\omega)$

$$\left[\begin{array}{l} A_1 = \{\omega_1^1, \omega_2^1, \cdots\} \\ A_2 = \{\omega_1^2, \omega_2^2, \cdots\} \\ \cdots \\ A_n = \{\omega_1^n, \omega_2^n, \cdots\} \end{array}\right]$$

$= \mathbb{P}(A_1) + \mathbb{P}(A_2) + \cdots + \mathbb{P}(A_n).$

Another proof of (1): Observe that ϕ and ϕ are mutually exclusive.

Therefore, $\mathbb{P}(\phi \cup \phi) = \mathbb{P}(\phi) + \mathbb{P}(\phi) \Rightarrow \mathbb{P}(\phi) = 2\,\mathbb{P}(\phi) \Rightarrow \mathbb{P}(\phi) = 0.$

Theorem: **(Inclusion-exclusion principle)**

Let $A_1, A_2, \dots, A_n$ be any events. Then

$$P(A_1 \cup A_2 \cup \cdots \cup A_n) = \sum_{1 \le i \le n} P(A_i) - \sum_{1 \le i_1 < i_2 \le n} P(A_{i_1} \cap A_{i_2})$$

$$+ \sum_{1 \le i_1 < i_2 < i_3 \le n} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) - \cdots$$

$$+ (-1)^{n+1} P(A_1 \cap A_2 \cap \cdots \cap A_n)$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, i_2, \dots, i_j \in \{1, 2, \dots, n\}}} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j})$$

For $n=3$

$$P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$

Remark: $A_1 = E$ $\quad A_2 = E^c$

$$\underset{\displaystyle P(\Omega)=1}{\underset{\|}{P(E \cup E^c)}} = P(E) + P(E^c) - \underbrace{P(E \cap E^c)}_{= P(\emptyset) = 0}$$

$$1 = P(E) + P(E^c) \Rightarrow P(E^c) = 1 - P(E).$$

Example: Suppose n people get their hats returned in random order. (different people, different hats). What is the chance that at least one person gets the correct hat?

- Here the cardinality of the sample space is $n!$.

Let A_i be the event that the person i gets the correct hat, for $1 \le i \le n$.

$$P(A_i) = \frac{1 \times (n-1)!}{n!} = \frac{1}{n} \quad \forall \; 1 \le i \le n.$$

$$P(A_i \cap A_j) = \frac{1 \times 1 \times (n-2)!}{n!} = \frac{1}{n(n-1)} \quad \forall \; 1 \le i \ne j \le n$$

$\cdots$

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = \frac{(n-k)!}{n!}$$

$\forall$ distinct $i_1, i_2, \ldots, i_k$

$i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$.

$2 \neq 3 \neq 2$

Our required event $\bigcup_{i=1}^{n} A_i$ has probability

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, \ldots, i_j \in \{1, 2, \ldots, n\}}} P(A_{i_1} \cap \cdots \cap A_{i_j})$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, \ldots, i_j \in \{1, 2, \ldots, n\}}} \frac{(n-j)!}{n!}$$

$$= \sum_{j=1}^{n} (-1)^{j+1} \binom{n}{j} \frac{(n-j)!}{n!} = \sum_{j=1}^{n} (-1)^{j+1} \frac{1}{j!}$$

<u>Example</u>: If 10 identical balls are thrown into 4 distinct boxes, calculate the probability that no box is empty.

$\Rightarrow$ Let us define the events:

$A_i = \{$ the ith box is empty $\}$ $\forall$ $i = 1, 2, 3, 4$.

Then our required event is $A_1^c \cap A_2^c \cap A_3^c \cap A_4^c = (A_1 \cup A_2 \cup A_3 \cup A_4)^c$

$$P(A_i) = \frac{\binom{10+3-1}{3-1}}{\binom{10+4-1}{4-1}} \quad \forall \ i = 1, 2, 3, 4.$$

$$\binom{n+K-1}{K-1}$$

$$P(A_i \cap A_j) = \frac{\binom{10+2-1}{2-1}}{\binom{10+4-1}{4-1}} \quad \forall\ i \neq j \text{ and } i,j \in \{1,2,3,4\}.$$

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1$$

$$P(A_i \cap A_j \cap A_k) = \frac{\binom{10+1-1}{1-1}}{\binom{10+4-1}{4-1}}$$

$$P(A_1 \cap A_2 \cap A_3 \cap A_4) = P(\phi) = 0.$$

Our required probability is

$$1 - P(A_1 \cup A_2 \cup A_3 \cup A_4) = 1 - \sum P(A_i) + \sum P(A_i \cap A_j) - \sum P(A_i \cap A_j \cap A_k) + P(A_1 \cap A_2 \cap A_3 \cap A_4) \quad = \frac{92}{143}$$

Proof of principle of inclusion-excusion:

Statement: $$\mathbb{P}\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, \ldots, i_j \in \{1,2,\ldots,n\}}} \mathbb{P}(A_{i_1} \cap \cdots \cap A_{i_j}).$$

We shall prove the statement by the method of induction.

For $n=2$, $\mathbb{P}(A_1 \cup A_2) = \mathbb{P}(A_1) + \mathbb{P}(A_2) - \mathbb{P}(A_1 \cap A_2)$

Observe that $\underline{A_1}$ and $\underline{A_2 \setminus A_1}\,(= A_2 \cap A_1^c)$ are mutually exclusive events (i.e. $A_1 \cap (A_2 \setminus A_1) = \phi$) and their union is $A_1 \cup A_2$.

Applying part (3) of Theorem 1 we get,

$$P(A_1 \cup A_2) = P(A_1 \cup (A_2 \setminus A_1)) = \underline{P(A_1) + P(A_2 \setminus A_1)} \quad \text{—— (1)}$$

Again, $A_2 \setminus A_1$ and $A_1 \cap A_2$ are mutually exclusive events and their union is A_2.

Applying part (3) of theorem 1 we get,

$$P(A_2) = P((A_2 \setminus A_1) \cup (A_1 \cap A_2)) = \underline{P(A_2 \setminus A_1) + P(A_1 \cap A_2)} \quad \text{—— (2)}$$

substracting equation (2) from equation (1) we get,

$$P(A_1 \cup A_2) - P(A_2) = P(A_1) - P(A_1 \cap A_2)$$

$$\Rightarrow \underline{P}(A_1 \cup A_2) = \underline{P}(A_1) + \underline{P}(A_2) - \underline{P}(A_1 \cap A_2).$$

Next day we shall finish the proof.