Probability theory 1

1/10/21

Theorem: (Inclusion exclusion principle)

Let $A_1, A_2, \cdots, A_n$ be events. Then

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1 < i_2 < \cdots < i_j \\ i_1, \cdots, i_j \in \{1, 2, \cdots, n\}}} P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_j}).$$

Pf: We proved for $n=2$. Let the statement be true for $n=k$.

Now suppose $A_1, \cdots, A_{k+1}$ be $k+1$ events. Then,

$$P\left(\bigcup_{i=1}^{k+1} A_i\right) = P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cup A_{k+1}\right)$$

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\left(\bigcup_{i=1}^{k} A_i\right) \cap A_{k+1}\right)$$

(induction hyp. for $n=2$)

$$= P\left(\bigcup_{i=1}^{k} A_i\right) + P(A_{k+1}) - P\left(\bigcup_{i=1}^{k} (A_i \cap A_{k+1})\right)$$

(distributive prop.)

$$= \boxed{\sum_{i=1}^{k} P(A_i)} - \sum_{1\le i_1<i_2\le k} P(A_{i_1} \cap A_{i_2}) + \sum_{1\le i_1<i_2<i_3\le k} P(A_{i_1} \cap A_{i_2} \cap A_{i_3})$$

$$- \cdots + (-1)^{k+1} P(A_1 \cap A_2 \cap \cdots \cap A_k)$$

$$\boxed{+ P(A_{k+1})}$$

$$- \sum_{i=1}^{k} P(A_i \cap A_{k+1}) + \sum_{1 \le i_1 < i_2 \le k} P(A_{i_1} \cap A_{i_2} \cap A_{k+1})$$

$$- \cdots - (-1)^k \sum_{1 \le i_1 < i_2 < \cdots < i_{k-1} \le k} P(A_{i_1} \cap \cdots \cap A_{i_{k-1}} \cap A_{k+1})$$

$$- (-1)^{k+1} P(A_1 \cap \cdots \cap A_k \cap A_{k+1})$$

(used induction hyp. for $n=k$ twice)

$$= \boxed{\sum_{i=1}^{k+1} P(A_i)} - \sum_{1 \le i_1 < i_2 \le k+1} P(A_{i_1} \cap A_{i_2}) + \sum_{1 \le i_1 < i_2 < i_3 \le k+1} P(A_{i_1} \cap A_{i_2} \cap A_{i_3}) -$$

$$\cdots + (-1)^{k+2}\, \mathbb{P}(A_1 \cap \cdots \cap A_{k+1}). \quad \square$$

• Let us see two inequalities.

(1) $\mathbb{P}(A \cup B) \leq \mathbb{P}(A) + \mathbb{P}(B)$ for two events A and B.

Pf: We have from the inclusion exclusion principle

$$\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$$

$$\leq \mathbb{P}(A) + \mathbb{P}(B) \quad (\text{since } \mathbb{P}(A \cap B) \geq 0).$$

(2) Bonferroni's inequality:

$$P(A\cup B) \geq P(A) + P(B) - 1$$ for two events A and B.

pf: From the inclusion exclusion principle, we have,

$$P(A) + P(B) = P(A\cup B) + P(A\cap B)$$

$$\leq P(A\cup B) + 1 \quad (\text{since } P(A\cap B) \leq 1)$$

$$\Rightarrow P(A\cup B) \geq P(A) + P(B) - 1.$$

$$\cancel{P(A\cup B) \geq P(A)+P(B)}$$

similarly, $P(A\cap B) \geq P(A) + P(B) - 1.$

Conditional probability:

Suppose we have data set

Persons	married	happy
Bipasha	yes	no
Palsh	Yes	Yes
Saptami	no	yes
Ali	no	no
...	...	...

total married persons

Married ↓ \ Happy →	Yes	no	
Yes	40	25	65
no	8	27	35
	48	52	100

total happy persons.

We perform a random experiment: pick a person from the above data set randomly.

Here the sample space Ω is the set of all persons and $\#\Omega = 100$.

Let us consider the following events:

$H = \{$ the chosen person is happy $\}$

$M = \{$ the chosen person is married $\}$

We know $\mathbb{P}(H) = \frac{48}{100}$ and $\mathbb{P}(M) = \frac{65}{100}$.

The probability that the chosen person is happy given that the person is married $= \dfrac{40}{65}$

$$P(H \mid M) = \frac{\frac{40}{100}}{\frac{65}{100}}$$

$\frac{40}{100} = P(H \cap M)$

$\frac{65}{100} = P(M)$

<u>Definition</u>: For events A and B with $P(B) > 0$, the conditional probability of A, given B has occured is defined by

$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}.$$

Example: A number is chosen at random from the first 100 natural numbers. ($x \in \{1, 2, \dots, 100\}$). Given the digit in the ten's place is 2, what is the conditional probability that the number is prime?

$\Rightarrow$ Intutively $\frac{2}{10}$ is the required probability.

20
21
2
23 →
4
:
29 →

10
45
ten's place.

Let us define the following events:

$A = \{$ the chosen number is prime $\}$

$B_i = \{$ the digit in the ten's place is i $\}$ for $i = 0, 1, 2, \dots, 9$

We need to compute $P(A|B_2) = \dfrac{P(A \cap B_2)}{P(B_2)}$

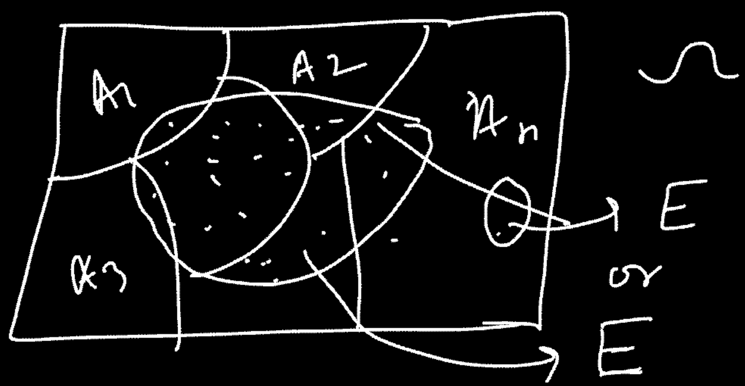
$$= \frac{\frac{2}{100}}{\frac{1}{10}} = \frac{2}{10} = \frac{1}{5}.$$

Now we shall state and prove the laws of total probability:

<u>Theorem 2:</u> If $A_1, A_2, \cdots, A_n$ are mutually exclusive and exhaustive events with positive probabilities, then for any event E, we have

$$P(E) = \sum_{i=1}^{n} P(E|A_i)\, P(A_i)$$

<u>Pf:</u>



$$P(E) = \sum_{i=1}^{n} P(E \cap A_i)$$

$\cancel{P(A)} > 0$

$$\text{RHS} = \sum_{i=1}^{n} P(E|A_i)\, \underline{P}(A_i)$$

$$= \sum_{i=1}^{n} \frac{P(E \cap A_i)}{P(A_i)} P(A_i) \quad \text{(by definion)} \left(\underline{P(A_i) > 0} \right)$$

$$= \sum_{i=1}^{n} P(E \cap A_i)$$

applied Theorem I (3).

$$= \underline{P}\left(\bigcup_{i=1}^{n} (E \cap A_i) \right)$$

(observe that $E \cap A_1, E \cap A_2, \cdots, E \cap A_n$ are mutually exclusive since $A_1, \cdots, A_n$ are so).

$$= \underline{P}\left(E \cap \left(\bigcup_{i=1}^{n} A_i \right) \right)$$

(distributive prop.)

$\forall i \neq j$, $(E \cap A_i) \cap (E \cap A_j)$

$= E \cap A_i \cap A_j \subseteq A_i \cap A_j = \phi$

$= P(E \cap \Omega)$ (since A_i's are exhaustive i.e. $\bigcup_{i=1}^{n} A_i = \Omega$)

$= P(E)$ $(\because E \subseteq \Omega)$.

$=$ LHS.

- $$P(E) = \sum_{i=1}^{n} P(E|A_i)\, P(A_i)$$

<u>Example:</u> Consider an urn with 10 white balls and 10 red balls. Two balls are drawn without replacement. (distinct balls)

(a) Given that the first ball drawn is red, what is the conditional probability that the second draw is also a red?

(b) Find the unconditional probability of the second ball drawn is being red?

$\Rightarrow$) (a) Let us define the following events :

$A_r = \{$ the first ball drawn is red $\}$

$A_w = \{$ " " " " " white $\}$

$B_r = \{$ " second " " " red $\}$

$B_w = \{$ " " " " " white $\}$

We want $P(B_r \mid A_r) = \dfrac{P(A_r \cap B_r)}{P(A_r)} = \dfrac{\frac{10 \times 9}{20 \times 19}}{\frac{10 \times 19}{20 \times 19}} = \dfrac{9}{19}$

(b) $P(B_r) = P(B_r \mid A_r) P(A_r) + P(B_r \mid A_w) P(A_w)$

$= \dfrac{9}{19} \times \dfrac{10}{20} + \dfrac{P(B_r \cap A_w)}{P(A_w)} \; P(A_w)$

$$= \frac{9}{19} \times \frac{10}{20} + \frac{10 \times 10}{20 \times 19} = \frac{9}{38} + \frac{10}{38} = \frac{19}{38} = \frac{1}{2}.$$