Probability theory 1

5/10/2021

- Last day we completed the proof of inclusion-exclusion principle.
- We proved the following inequalities:

$$P(A \cup B) \leq P(A) + P(B)$$

is called Boole's inequality (or Bonferroni's inequality)

$$P(A \cup B) \geq P(A) + P(B) - 1$$

$$\boxed{P(A) + P(B) - 1 \leq P(A \cup B) \leq P(A) + P(B)}$$

- We had an introduction to conditional probability.

- Recall : Theorem:

If $A_1, \dots, A_n$ are mutually exclusive and exhaustive events with positive probabilities, then for any event E, we have,

$$P(E) = \sum_{i=1}^{n} P(E|A_i)\, P(A_i)$$

(positive probabilities needed)

Rough:

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{j=1}^{n} (-1)^{j+1} \sum_{\substack{i_1<\cdots<i_j \\ i_1,\dots,i_j \in \{1,2,\dots,n\}}} \underbrace{P(A_{i_1} \cap \cdots \cap A_{i_j})}_{=S_j}$$

$$= \sum (-1)^{j+1} \sum S_j$$

If n is even

$$P\left(\bigcup_{i=1}^{n} A_i\right) \geq \sum_{j=1}^{K} (-1)^{j+1} \sum_{i_1<\cdots<i_j} S_j \qquad 1 \leq K \leq n$$

If n is odd

$$P\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{j=1}^{K} (-1)^{j+1} \sum_{i_1<\cdots<i_j} S_j \qquad 1 \leq K \leq n$$

- We saw in the final example on the last day that the probability of picking a ball in the second time was same as picking a ball in the first time. This phenomenon is called Polya urn scheme.

- Consider an urn containing W white balls and B black balls with $W+B \geq 2$. Pick k $(1 \leq k \leq W+B)$ balls from the urn, without replacement. We want to calculate the probability that the k^{th} ball drawn is white.

<u>Theorem</u>: If balls are drawn at random from an urn containing W white and B black balls (for any choice of W, B s.t. $W+B \geq 2$), then for all $1 \leq k \leq W+B$

$$P(\text{the } k\text{-th ball drawn is white}) = \frac{W}{W+B}$$

<u>Pf</u>: We shall prove this by the method of induction.

$$P(\text{the 1-st ball drawn is white}) = \frac{W}{W+B}$$

Suppose $P(\text{the } k\text{-th ball drawn is white}) = \dfrac{W}{W+B}$

for all $1 \le k \le n < W+B$ and for all W, B s.t. $W+B \ge 2$.

Now,

$P(\text{the } n+1 \text{ th ball drawn is white})$ $\quad \boxed{W_k = \{k\text{th draw is white}\}}$

$= P(\text{the } n+1^{\text{th}} \text{ ball drawn is white} \mid \text{the 1st draw is white})\, P(\text{the } 1^{\text{st}} \text{ draw is white})$

$+ P(\text{the } n+1\text{th draw is white} \mid \text{the } 1^{\text{st}} \text{ draw is black})\, P(\text{the } 1^{\text{st}} \text{ draw is black})$

$= \dfrac{W-1}{W+B-1} \dfrac{W}{W+B} + \dfrac{W}{W+B-1} \dfrac{B}{W+B}$ —— (*)

$\boxed{\begin{matrix} W-1 \\ B \end{matrix}}$ 1st case $\qquad \boxed{\begin{matrix} W \\ B-1 \end{matrix}}$ 2nd case rough:

$$\frac{P(n+1^{th}\ W \mid 1^{st}\ W)}{\boxed{W/B}} = \frac{P(n^{th}\ W)}{\boxed{W-1/B}}.$$

for any n it is true that

"for any integer its square is an integer"

$$\circledast = \frac{W^2 - W + WB}{(W+B-1)(W+B)} = \frac{W(\cancel{W+B-1})}{(W+B)(\cancel{W+B-1})} = \frac{W}{W+B} \quad \square$$

$$\Downarrow (W+B \geq 2)$$

Rough

$\underbrace{A}_{''} \Rightarrow$ A is true for $n=1$

Assume A is true for $n=k$ — (1)

prove that A is true for $n=k+1$ assuming (1).] A is true $\forall n \in \mathbb{N}$.

statement

$A \Rightarrow$ $\left[\begin{array}{l} \text{A is true for } K=1 \; (/ n=1) \\ \text{Assume A is true for all } K=1,2,\dots,n \; (1 \le K \le n) \text{ — ⓐ} \quad (n=1,2,\dots,K) \\ \text{prove that A is true for } K=n+1 \text{ assuming ⓐ.} \\ \qquad (\text{or } n=K+1) \end{array}\right]$ A is true $\forall n \in \mathbb{N}$.

$\parallel$

for any $w>0$, ...

A ...

- We assumed 'A' is a statement. Don't confuse with event A.
- These are the two ways to prove a statement by the method of induction on the set of natural numbers.

Bayes' theorem:

Suppose $A, B_1, \cdots, B_n$ are events such that

$P(A) > 0$ and $P(B_i) > 0$ for all $i \in \{1, 2, \cdots, n\}$.

Assume that $B_1, B_2, \cdots, B_n$'s are mutually exclusive and exhaustive.

Then,

$$P(B_i | A) = \frac{P(A|B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A|B_j)\, P(B_j)}.$$

Pf: $\text{LHS} = P(B_i | A) = \dfrac{P(B_i \cap A)}{P(A)}$ (by definition)

$$= \frac{\frac{P(B_i \cap A)}{P(B_i)} P(B_i)}{P(A \cap \Omega)} = \frac{P(A|B_i)\, P(B_i)}{P\left(A \cap \left(\bigcup_{j=1}^{n} B_j\right)\right)}$$

(distributive prop.)

$$= \frac{P(A|B_i)\, P(B_i)}{P\left(\bigcup_{j=1}^{n} (A \cap B_j)\right)}$$

($A \cap B_i$'s are mutually exclusive).

$$= \frac{P(A|B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A \cap B_j)}$$

$$= \frac{P(A|B_i)\, P(B_i)}{\sum_{j=1}^{n} P(A|B_j)\, P(B_j)} \qquad \square$$

Example: Two balls are drawn from an urn containing W white balls and B black balls. Given the second ball is black what is the conditional probability that the first drawn ball is white?

- Let us define the following events:

$$W_i = \{\text{the ith drawn ball is white}\} \text{ for } i=1,2.$$

$$B_i = \{\text{the ith drawn ball is black}\} \text{ for } i=1,2.$$

We need $P(W_1 \mid B_2) = \dfrac{P(B_2 \mid W_1)\, P(W_1)}{\{P(B_2 \mid W_1)\, P(W_1) + P(B_2 \mid B_1)\, P(B_1)\}}$

$P(B_2) = \{P(B_2 \mid W_1)\, P(W_1) + P(B_2 \mid B_1)\, P(B_1)\}$

$$= \frac{\frac{B}{W+B-1} \cdot \frac{W}{W+B}}{\frac{B}{W+B-1} \cdot \frac{W}{W+B} + \frac{B-1}{W+B-1} \cdot \frac{B}{W+B}}$$

$$\frac{BW}{BW+B^2-B}$$

$$= \frac{W}{W+B-1}$$

$$= \frac{BW}{(W+B)(W+B-1)} \cdot \frac{W+B}{B}$$

$$= \frac{W}{W+B-1} \quad \square$$