Probability theory 1

6/10/21

- Yesterday we saw the Polya urn scheme and the Bayes' theorem. We discussed some examples. We continue with another example.

- <u>Example</u>: Consider a multiple choice test where each question has 4 options and exactly one correct answer. A participant knows the correct answers to a particular question with probability 0·8. If the student doesn't know the correct answers, she decides to make a random choice. Given that she has answered a question correctly, what is the conditional probability that she knew the answer?

→ Let us define the following events.

$A = \{ \text{she knows the answer} \}$, $P(A) = \frac{8}{10}$

$B_i = \{ \text{she choses option } i \}$
$i = 1, 2, 3, 4$

We need to find $P(A | B_1)$ where we suppose that option

1 is the right answer to that particular question. Now applying Bayes' theorem we get

$$P(A|B_1) = \frac{P(B_1|A)\,P(A)}{P(B_1|A)\,P(A) + P(B_1|A^c)\,P(A^c)}$$

$$= \frac{1 \times \frac{8}{10}}{\frac{8}{10} + \frac{1}{4} \times \frac{2}{10}} = \frac{8}{8+0.5} = \frac{16}{17} \quad \square$$

Independent events:

Consider two events A and B with positive probabilities. Suppose that the conditional probability of A given B (i.e. $P(A|B)$) is same as the unconditional probability of A (i.e. $P(A)$)

$$P(A|B) = P(A) \Rightarrow \frac{P(A\cap B)}{P(B)} = P(A)$$

$$\Rightarrow \boxed{P(A\cap B) = P(A)\,P(B)}$$

Similarly, $\underline{P(B|A) = P(B)} \Rightarrow \boxed{P(A \cap B) = P(A)\,P(B).}$

Definition: For two events A and B, we call them to be independent if $P(A \cap B) = P(A)\,P(B)$.

Remark 1: A and B or both may be empty set or events with zero probability.

Remark 2: Disjointness should not be confused with independence.

Examples: Let a fair dice be rolled two times.

Let us define the following events:

$E = \{ (2,2), (4,4), (6,6) \text{ are the outcomes} \}$

$\theta = \{ \text{the outcomes are } (1,1), (3,3), (5,5) \}$

Then $p(E) = \frac{3}{36} = p(\theta)$

$p(E \cap \theta) = p(\phi) = 0 \neq \left(\frac{3}{36}\right)^2 = p(E)\, p(\theta)$

E and θ are disjoint and they are not independent.

Let $\theta_1 = \{$ the outcome is $(1,1)\}$

$$P(\theta \cap \theta_1) = P(\theta_1) = \frac{1}{36}$$

$$P(\theta)\,P(\theta_1) = \frac{3}{36} \cdot \frac{1}{36} \neq P(\theta \cap \theta_1)$$

$$\underset{\substack{\| \\ 0}}{P(A \cap B)} = \underset{\substack{\uparrow \\ 0}}{P(A)}\,\underset{\substack{\uparrow \\ +ve}}{P(B)}$$

θ and θ_1 are not disjoint and they are not independent.

Let $A = \{$ the first outcome is $1\}$

$B = \{$ the second outcome is $2\}$

$$P(A \cap B) = \frac{1}{36}$$

$$P(A) = \frac{1}{6} = P(B).$$

$P(A|B) = P(A)$, $P(B|A) = P(B)$.

$$\Downarrow$$

$$P(A \cap B) = P(A)\,P(B).$$

A and B are not disjoint but A and B are independent.

<u>Fact:</u> If A and B are disjoint and they are independent then, at least one of the events from A or B (i.e. either $P(A)$ or $P(B)$ or both) must be equal to zero.

<u>Example:</u> A card is chosen at random from a deck of playing cards. Let

$S = \{$ the drawn card is a spade $\}$

$A = \{$ the drawn card is an ace $\}$. Then,

$$P(S) = \frac{13}{52} = \frac{1}{4} \quad \text{and} \quad P(A) = \frac{4}{52} = \frac{1}{13}.$$

$$P(S \cap A) = P(\{\text{the drawn card is ace of spades}\})$$

$$= \frac{1}{52} = \frac{1}{4} \cdot \frac{1}{13} = P(A)\, P(S).$$

So, A and S are independent.

Theorem: Suppose A and B are two independent events. Then the following events are also independent:

(1) A and B^c

(2) A^c and B

(3) A^c and B^c

Proof: We need to show $P(A \cap B^c) = P(A)\,P(B^c)$ and we have

$$P(A \cap B) = P(A)\,P(B)$$

$$\text{LHS} = P(A \cap B^c) = P(A \cap (\Omega - B)) = P(A \setminus B) = P(A \setminus (A \cap B))$$

$$= P(A) - P(A \cap B)$$

Since, $A \cap B$, $A \cap B^c$ are exclusive and

$= P(A) - P(A)P(B)$ ($\because$ A and B are independent)

$= P(A)\left(1 - P(B)\right)$

$= P(A)\,P(B^c)$ (Remark of Theorem (1) part (3)).

$=$ RHS.

$(A \cap B) \cup (A \cap B^c) = A$

$\Downarrow$

$P(A \cap B) + P(A \cup B^c) = P(A)$

(Theorem (1) part (3))

(2) follows from similar computations done in part (1).

(3) A and B are independent $\Rightarrow$ A^c and B are independent (by part (2))

$\Rightarrow$ B^c and A^c are independent (by part (2)). $\square$