Probability theory 1

8/10/21

- Last day we saw the definition of independence for two events and saw some examples. We continue with another example.
- Example: Consider the six permutations of the letters a, b, and c as well as the three triples aaa, bbb and ccc i.e. the sample space is

$$\Omega = \{ \underset{\uparrow}{abc}, \underset{\uparrow}{acb}, bac, bca, cab, cba, \underset{\uparrow}{\underline{aaa}}, bbb, ccc \}.$$

We pick a member of Ω with probability $\frac{1}{9}$.

We define, $A_k = \{$ the picked word has a in the k^{th} place$\}$

for $k = 1, 2, 3$.

Then, $P(A_1) = \frac{3}{9} = \frac{1}{3} = P(A_2) = P(A_3)$.

Also, $P(A_1 \cap A_2) = \frac{1}{9} = P(A_1 \cap A_3) = P(A_2 \cap A_3)$

$$\begin{array}{ccc} P(A_1 \cap A_2) & P(A_1 \cap A_3) & P(A_2 \cap A_3) \\ \| & \| & \| \\ P(A_1)\,P(A_2) & P(A_1)\,P(A_3) & P(A_2)\,P(A_3) \end{array}$$

Thus we have A_i and A_j are independent for $i \neq j$ and $i, j \in \{1,2,3\}$.

$$P(A_1 \cap A_2 \cap A_3) = \frac{1}{9} \neq \left(\frac{1}{3}\right)^3 = P(A_1)\,P(A_2)\,P(A_3)$$

We shall say in this case that A_1, A_2, and A_3 are not independent but they are pairwise independent.

· We want to define independence for more that two events.

<u>Definition:</u>

The events $A_1, A_2, \cdots, A_n$ are called independent/ mutually independent if for all combinations $1 \le i < j < k < \cdots \le n$ the multiplication rules

$$\left.\begin{aligned} P(A_i \cap A_j) &= P(A_i)\,P(A_j) \\ P(A_i \cap A_j \cap A_k) &= P(A_i)\,P(A_j)\,P(A_k) \\ &\cdots \\ &\cdots \\ P(A_1 \cap A_2 \cap \cdots \cap A_n) &= P(A_1)\,P(A_2)\cdots P(A_n) \end{aligned}\right\} \circledast$$

holds true.

. The first line contains $\binom{n}{2}$ equations, the second line contains $\binom{n}{3}$ equations and so on. So there are

$$\binom{n}{2} + \binom{n}{3} + \cdots + \binom{n}{n} = (1+1)^n - \binom{n}{0} - \binom{n}{1} = 2^n - n - 1$$

Conditions which needs to be satified. Observe that the first $\binom{n}{2}$ conditions suffiees to ensure that the events $A_1, A_2, \cdots, A_n$ are pairwise independent.

$$\circledast \left\{ \begin{array}{l} \phi \neq \Lambda \subseteq \{1, 2, \cdots, n\} \\ P\left(\bigcap_{\alpha \in \Lambda} A_\alpha \right) = \prod_{\alpha \in \Lambda} P(A_\alpha) \end{array} \right.$$

short form

Theorem: If $A_1, A_2, \cdots, A_n$ are independent then $A_{i_1}, A_{i_2}, \cdots, A_{i_k}$ are also independent for distinct choices of the indices $i_1, i_2, \cdots, i_k \in \{1, 2, \cdots, n\}$.

Pf: This follows from the definition of independence.

Example: A fair coin is tossed twice. Define the events

$A = \{\text{The first toss is a Head}\}$

$B = \{\text{The second toss is a Head}\}$

$C = \{\text{Exactly one of the tosses is a Head}\}$

Then, A, B, and C are pairwise independent but not independent.

Observe that $C = A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Example: A number is chosen at random from $\{0, 1, 2, \cdots, 10^n - 1\}$. For $i = 1, 2, \ldots, n$, let A_i be the event that the i^{th} digit from the right is even. Then the events $A_1, A_2, \dots, A_n$ are independent.

$$\Rightarrow \quad P(A_1) = \frac{5}{10} = \frac{1}{2} = P(A_i) \quad \forall\, i \in \{2, \dots, n\}$$

$$P(A_i \cap A_j) = \frac{5 \times 5}{10 \times 10} = \frac{5}{10} \times \frac{5}{10} = P(A_i)P(A_j).$$

$\dots$

For any $2 \le k \le n$

$$P(A_{i_1} \cap \dots \cap A_{i_k}) = \left(\frac{1}{2}\right)^k = P(A_{i_1}) \cdots P(A_{i_k}).$$

$\Omega = \{0, 1, \dots, 10^n - 1\}$

$a_0 a_1 a_2 \dots a_{n-1} a_n$ → 2nd place

$a_i \in \{0, \dots, 9\}$

$7\ 6\ 2\ \dots\ 5$ ← 1st place

$0\ 0\ 0\ 0\ 1\ 0$

$$\frac{5 \times 10^{n-1}}{10^n}$$

$$P(A_1) = \frac{4 \times 10^{n-1}}{10^n - 1}$$ if 0 is excluded.

if $\Omega = \{1, \dots, 10^n - 1\}$

<u>Theorem:</u> Suppose that $A_1, A_2, \cdots, A_n$ are independent events. If $i_1, i_2, \cdots, i_k, j_1, j_2, \cdots, j_l \in \{1, 2, \cdots, n\}$ are distinct, then

$$P\left(A_{i_1} \cap \cdots \cap A_{i_k} \cap A_{j_1}^c \cap A_{j_2}^c \cap \cdots \cap A_{j_l}^c\right) = P(A_{i_1}) \cdots P(A_{i_k}) P(A_{j_1}^c) P(A_{j_2}^c) \cdots P(A_{j_l}^c)$$

<u>Pf:</u> Understanding the statement:

Suppose A_1, A_2, A_3, A_4, A_5 are independent. $\Rightarrow$ A_1, A_2, A_3 are idependent

The theorem implies $P(A_4 \cap A_2 \cap A_5^c \cap A_1^c) = P(A_4) P(A_2) P(A_5^c) P(A_1^c)$

$$P(A_1 \cap A_2^c \cap A_3^c) = P(A_1) P(A_2^c) P(A_3^c)$$

A, B ind.
$\Rightarrow A, B^c$ ind.

It is enough to prove that $P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots A_{k+l}^c)$

$$= P(A_1)\cdots P(A_k)\,P(A_{k+1}^c)\cdots P(A_{k+l}^c) \quad \begin{pmatrix} k+l \le n \\ k, l \geq 1 \end{pmatrix}$$

because we may rename the A_{i_r}'s and A_{j_s}'s by $A_1, \dots, A_k$ and $A_{k+1}, \dots, A_{k+l}$

We shall prove this by induction.

Fix a $k \in \{1, 2, \dots, n\}$. We shall do the induction on l.

For $l = 1$, we need to show $P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c) = P(B)\,P(A_{k+1}^c)$ (where $B = A_1 \cap \cdots \cap A_k$)

The above is true because $\underline{A_1, \dots, A_{k+1} \text{ are independent}}$

and therefore $\underline{(A_1 \cap \cdots \cap A_k)}$ and $\underline{A_{k+1} \text{ are independent}}$

and therefore $(A_1 \cap \cdots \cap A_k)$ and A_{k+1}^c are independent.

(by previous day's theorem)

rough

$P(B \cap A_{k+1}) \overset{?}{=} P(B)\,P(A_{k+1})$

$P(B \cap A_{k+1}) = P(A_1 \cap \cdots \cap A_k \cap A_{k+1})$

$= P(A_1) \cdots P(A_{k+1})$

$= P(A_1 \cap \cdots \cap A_k)\,P(A_{k+1})$

$= P(B)\,P(A_{k+1})$

Suppose the statement is true for $l = q$.

Now we have $\underline{P(A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_{k+q}^c) = P(A_1) \cdots P(A_k) P(A_{k+1}^c) \cdots P(A_{k+q}^c)}$

$$P(\underbrace{A_1 \cap \cdots \cap A_k \cap A_{k+1}^c \cap \cdots \cap A_{k+q}^c}_{=B_1} \cap A_{k+q+1}^c) =$$

. postpone for now.

Example: Consider the polya urn scheme with w white balls and b black balls. Suppose that after each draw, a total of c balls having the same colour of the drawn ball are added to the urn. Calculate the probability that the n-th drawn ball is white.

$\Rightarrow$ Let $W_i = \{\text{The ith draw is white}\}$ $\forall\ i \geq 1$

$B_i = \{\text{The ith draw is black}\}$ $\forall\ i \geq 1$

$$P(W_1) = \frac{w}{w+b}$$

$$p(W_2) = p(W_2 \mid B_1)\, p(B_1) + p(W_2 \mid W_1)\, p(W_1)$$

$$= \frac{w}{\underbrace{w}_{}+\underbrace{b+c-1}_{}} \; \frac{b}{w+b} + \frac{w+c-1}{b+w+c-1} \; \frac{w}{w+b}$$

$$= \frac{w\,(b+w+c-1)}{(w+b)\,(w+b+c-1)} = \frac{w}{w+b} .$$

This is the induction step for $n = 2$.

Finish the induction step by conditioning on the first draw.

Problem! Suppose that A and B are events with $0 < p(B) < 1$

and $\underline{P(A|B) = P(A|B^c)}$. Show that A and B are independent.

Ans:

$P(A|B)$

$\parallel$

LHS $\dfrac{P(A\cap B)}{P(B)}$

$P(A|B^c)$

$= \dfrac{P(A\cap B^c)}{P(B^c)} = \dfrac{P(A\cap B^c)}{1-P(B)}$

$= \dfrac{P(A\setminus(A\cap B))}{1-P(B)} = \dfrac{P(A)-P(A\cap B)}{1-P(B)}$.

$\underbrace{\underbrace{A\cap B}, \underbrace{A\setminus(A\cap B)}}_{A}$

$\underbrace{P(A\cap B)} - \cancel{P(A\cap B)\,P(B)} \overset{?}{=} \underbrace{P(A)P(B)} - \cancel{P(B)\,P(A\cap B)}$.

$\Downarrow$

$P(A\cap B) = P(A)\,P(B)$.

Another soln.:

$$P(B|A) = \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B^c)\,P(B^c)} \quad (\text{Bayes' theorem})$$

If $P(A) \neq 0$.

$$= \frac{P(A|B)\,P(B)}{P(A|B)\,P(B) + P(A|B)\,(1 - P(B))} \quad \left(\because \underset{\text{given}}{P(A|B^c) = P(A|B)}\right)$$

$$= \frac{P(A|B)\,P(B)}{P(B)\,P(A|B) + P(A|B) - P(A|B)\,P(B)} = \frac{P(A|B)\,P(B)}{P(A|B)}$$

$$= P(B). \quad \Rightarrow A \text{ and } B \text{ and independent.} \quad \square.$$

$\Rightarrow P(A \cap B) = P(A)\,P(B)$

(If $P(A) = 0$ then A, B are independent trivially).

Next class 20/10/21 (Wedenesday) (by Arijit sir).